

Existence and Uniqueness of Mild Solutions for Neutral Nonlinear Fractional Evolution Systems with Integral– Multipoint Nonlocal Boundary Conditions

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ABSTRACT

Fractional differential evolution systems provide a fundamental framework for modeling processes with memory and hereditary effects in infinite-dimensional settings. Although substantial progress has been achieved in establishing existence results for nonlinear fractional systems with nonlocal boundary conditions, most available studies treat non-neutral models and restrict boundary interactions to scalar or constant-coefficient formulations. The analysis of neutral fractional evolution systems coupled with operator-valued integral–multipoint nonlocal boundary functionals remains comparatively underdeveloped, particularly in abstract Banach spaces where boundary operators interact nonlinearly with neutral dynamics. Motivated by this gap, we introduce a class of neutral nonlinear fractional evolution systems governed by the Caputo derivative of order $0 < \alpha < 1$, where the linear component generates a sectorial operator. The boundary condition is described by a nonlinear operator-valued functional combining integral terms and multipoint evaluations, thereby inducing a strongly coupled global interaction between interior evolution and boundary constraints. By constructing the associated fractional resolvent family, the system is reformulated as an equivalent operator equation in an appropriate function space. Existence of mild solutions is established using a measure-of-noncompactness approach together with Sadovskii's fixed-point theorem, while uniqueness is derived under generalized growth conditions without imposing restrictive global Lipschitz assumptions. The obtained results extend several known solvability criteria for non-neutral fractional systems and classical nonlocal boundary conditions. A MATLAB-based computational example is provided to illustrate the applicability of the theoretical framework and to support the uniqueness of solutions

Index Terms— *Neutral fractional evolution systems; Operator-valued nonlocal boundary functional; Mild solutions; Measure of noncompactness; Fixed-point theory*

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I. INTRODUCTION

Fractional evolution equations give a basic analytical approach of studying memory-dependent processes in unlimited dimensions. Such systems are the natural ones that are studied in the context of Banach space, and their approach to nonlinear problems with Caputo derivatives of order $0 < \alpha < 1$, is by using the theory of sectorial operators and fractional resolvent families. During the last ten years, the existence results of nonlinear fractional systems subject to nonlocal boundary conditions, such as multipoint and integral formulations, that are solved using fixed-point methods have been found to significant progress.

Non-neutral systems and non-neutral boundary conditions with scalar or constant coefficients are the subjects of the majority of the available contributions in spite of these developments. Further complexity is added to the analytical treatment of neutral fractional evolution systems in which the fractional derivative is taken on a functional representation of the unknown state. In case of such neutral dynamics, coupled with operator valued integral multipoint boundary functionals, the resulting system has a very appealing interaction of interior evolution and global

boundary constraints. This is a rather under-explored setting, especially in an abstract Banach space setting.

Driven by this gap, the current paper develops a family of neutral nonlinear fractional evolution equations, which are driven by the Caputo form of the derivative of order $0 < \alpha < 1$, in which the linear part forms a sectorial operator. The boundary condition is formulated in terms of an operator-valued functional of nonlinearity which brings together the contribution of integrals and multipoint evaluations hence the generalization of classical nonlocal boundary formulations. The main purpose is to create being and distinctiveness of mild solutions of this system in favorable growth and continuity circumstances. In order to accomplish this, the related fractional resolvent family is built and the issue is transformed into an equivalent operator equation in an appropriate function space. Existence is achieved through a measure-of-noncompactness method along with Sadovskii fixed-point theorem whereas uniqueness is achieved under generalized growth conditions without having global Lipschitz conditions. A computational example has been provided in Matlab to show the relevance of the theoretical framework.

The rest of the paper is constructed in the following way. Section 2 discussing the previous works regarding our goals, Section 3 illustrates the pre-conditions and working environment. Section 4 will be the formulation of the problem and the mild solution. The principal existence and uniqueness results are found in section 5. The numerical example is given in section 6.

II. LITERATURE REVIEW

In the past years, the nonlocal conditions have been studied widely in the fractional equations of evolution. The authors of the study Chen et al. [1] considered the fractional evolution equations, which are represented by nonlocal boundary conditions, through resolvent operator and controllability. The work on system of fractional evolution was extended to large extents by Chen and Zhou [2], who took existence and control into account in abstract operator settings.

Fractional systems have also been studied with respect to their dynamical behavior using synchronization and neural network models. Chowdhury et al. [3] considered the synchronizing effect of non-autonomous non-fractional order neural networks, Du and Lu [4] considered the finite time synchronization of the fractional order delayed fuzzy systems. Ghosh et al. [5] considered nonlinear wave structures of non-autonomous fractional models. These papers pay more attention to dynamism rather than the abstract solvability.

Han et al. [6], which took the null controllability of ordinary differential system, addressed the controllability problem of differential system and of the fractional system. Regularity results of non-autonomous equations of fractional evolution were given by him and Zhou [7] and this has been used to understand the analytic theory of fractional operators in Banach spaces. He and Lin [8] once attended the study of fractional Black-Scholes equations and explained their use in finance.

Further applications are in non-autonomous phase formation models [9] and controllability and Ulam stability of fractional damped evolution equations [10]. Li and Luo [11] studied stochastic fractional evolution equations and Li and Yong [12] provided a general framework to the infinite dimensional optimal control systems.

Lin and He [13] also examined the financial fractional models and Lu et al. [14] examined the solvability and optimal control of semilinear fractional evolution equations. Sugandha et al. [15] suggested analysis procedures of the fractional Black-Scholes equation and Surendar et al. analyzed the fractional hybrid nonlinear systems [16]. Balasubramaniam and Tamilalagan [17] discussed the Poisson jump fractional stochastic equations.

Wang et al. [18] developed abstract fractional Cauchy problems of almost sectorial operators in theory, which is the foundation of the analysis of fractional systems of evolution. Along with certain crucial outcomes of the abstract parabolic evolution equations, Yagi [19] defined some. Zhou [20] researched on attractivity properties of evolution equations of fractional equations and researched

on nonlinear non-autonomous evolution problems of fractional equations [21]. Other outdated existence results of fractional neutral evolution equations were obtained by Zhou and Jiao [22]. Zhu and Han [23] investigated approximate controllability of non-autonomous mixed type fractional equations.

Most of the former studies though contributing to the theory of fractional evolution are concerned with either controllability, synchronization, stochastic analysis or classical nonlocal conditions. Single integration of neutral fractional evolution systems with operator-valued nonlocal integral-multipoint boundary conditions in a measure-of-noncompactness framework has not been studied very far. The gap is addressed by the current paper that defines existence and uniqueness of mild solutions using the condensing operator methodology based on the fixed-point methods.

III. PRELIMINARIES

Assume that X is a real Banach space with norm $\|\cdot\|$. By $C([0, T], X)$ we mean the Banach space of all functions on the interval $[0, T]$ which are continuous into X having the supremum norm.

$$\|x\|_\infty = \sup_{t \in [0, T]} \|x(t)\| \quad (1)$$

This is because throughout this work, we assume that $0 < \alpha < 1$ that is, $0 < \alpha < 1$. The Caputo fractional derivative of order $0 < \alpha < 1$ of a continuous differentiable function $x: [0, T] \rightarrow X$ is given by,

$${}^c D_t^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} x'(s) ds, t \in (0, T] \quad (2)$$

Where and $\Gamma(\cdot)$ is the Gamma function. Caputo derivative is likely the most suitable derivative of the evolution equation as it allows classical initial conditions to be used in the fractional representation. Suppose $D(A) \subset X \rightarrow X$ is a closed linear operator, whose domain, $D(A)$ is dense. The operator A is called sectorial when there are constants $M > 0$ and $\theta \in (0, \pi)$ which are such that the resolvent has the estimate.

$$\|(\lambda I - A)^{-1}\| \leq \frac{M}{|\lambda|}, \lambda \notin \Sigma_\theta \quad (3)$$

Where,

$$\Sigma_\theta = \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leq \theta\} \quad (4)$$

When A is sectorial, then it produces a fractional resolvent family $\{S_\alpha(t)\}_{t \geq 0}$ which extends the classical semigroup to the fractional case. Specifically, there is a constant $M > 0$ satisfying such that:

$$\|S_\alpha(t)\| \leq M, t \geq 0 \quad (5)$$

The linear fractional evolution equation is given under this framework.

$${}^c D_t^\alpha x(t) = Ax(t), t \in (0, T] \quad (6)$$

Having the initial condition $x(0) = x_0 \in X$ has the mild solution representation.

$$x(t) = S_\alpha(t)x_0, t \in [0, T] \quad (7)$$

To avoid the use of stringent compactness requirements in the treatment of nonlinear problems in infinite-dimensional spaces, we recall the measure of noncompactness. Let B be a bounded subset of X . Kuratowski measure of the noncompactness of B is given by,

$$\chi(B) = \inf\{\varepsilon\} > 0: B \# 8$$

(Where B is covered by a finite number of sets of diameters not greater than epsilon). An operator $T: X \rightarrow X$ is said to be condensing when,

$$\chi(T(B)) < \chi(B) \quad (9)$$

where B is any bounded set of X and $\chi(B)$ above 0. Condensing operators generalize the concept of compact operators and permit the use of principles of generalized fixed-point. We will use the fixed-point theorem of Sadovskii, according to which given that D is a non-empty, closed, bounded and convex set of a Banach space E , and that $T: D \rightarrow D$ is a condensing and continuous map, then T has a fixed point in D . The above tools of analysis yield the functional platform that is required to determine existence and uniqueness of mild solutions of the nonlinear fractional evolution system of neutral operator-valued integral-multipoint boundary conditions that are addressed in the following sections.

IV. PROBLEM FORMULATION

In this part, we develop the neutral nonlinear fractional evolution system with operator valued integral-multipoint nonlocal boundary condition that shall be examined in the present work. Assume that X is a real Banach space, and $A: D(A) \subset X \rightarrow X$ is a sectorial operator that produces the family of fractional resolvents $\{S_\alpha(t)\}_{t \geq 0}$. We take the neutral fractional evolution equation as below:

$${}^c D_t^\alpha [x(t) - G(t, x(t))] = Ax(t) + F(t, x(t)), t \in (0, T] \quad (10)$$

where $0 < \alpha < 1$. The nonlinear mappings

$$F: [0, T] \times X \rightarrow X, G: [0, T] \times X \rightarrow X$$

The equation 10 denote the nonlinear dynamics within the internal perturbation, and the neutral perturbation, respectively.

The fact that the neutral term is $G(t, x(t))$ is a sign that the fractional derivative is applied to a functional form of the unknown solution, and not to the actual solution. This is the structural property that defines the difference between system (10) and classical non-neutral fractional systems by adding further analytical challenges in constructing mild solutions.

The operator-valued integral-multipoint nonlocal boundary condition is added to the system.

$$x(0) = \sum_{i=1}^m B_i(x(t_i)) + \int_0^T K(s, x(s))ds \quad (11)$$

where $0 < t_1 < t_2 < \dots < t_m < T$. Each mapping $B_i: X \rightarrow X$ is a nonlinear operator that defines multipoint interaction at the boundary, and

$K: [0, T] \times X \rightarrow X$ is nonlinear operator that describes the integral boundary contribution. In contrast to the classical multipoint conditions with scalar coefficients, the operators B_i can be nonlinearly dependent upon the state variable. Equally, the integral term is not limited by a linear functional but can be a nonlinear function of the solution. The boundary condition, in turn, makes the initial state globally coupled with the whole trajectory of the solution on $[0, T]$, which is far reaching the usual formulations of nonlocal boundary conditions. In order to study system (10)-(11), we transform it into a corresponding integral formulation. With the properties of the family of fractional resolvents of A , the neutral fractional equation (10) can be rewritten in mild form as.

$$x(t) = S_\alpha(t)(x(0) - G(0, x(0))) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) (F(s, x(s)) + AG(s, x(s)))ds, t \in [0, T] \quad (12)$$

Replacing the condition (11) into equation (12), we get the operator equivalent equation.

$$x(t) = S_\alpha(t) \left(\sum_{i=1}^m B_i(x(t_i)) + \int_0^T K(s, x(s))ds - G(0, x(0)) \right) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) (F(s, x(s)) + AG(s, x(s)))ds, \quad (13)$$

$$t \in [0, T]$$

Hence, a solution to the neutral fractional boundary value problem (10)-(11) is a fixed point of the operator $\mathcal{T}: C([0, T], X) \rightarrow C([0, T], X)$ defined by the right-hand side of equation (13).

To carry out the analysis further, we make the following structural assumptions:

(H1) Both F and G are continuous on $[0, T] \times X$. The two mappings grow well on $[0, T] \times X$ such that the integral operator in (12) is continuous and bounded on compact subsets of $([0, T], X)$.

(H2) $B_i: X \rightarrow X$ is continuous and bounded sets are mapped to bounded sets, $i = 1, 2, \dots, m$

(H3) This mapping $K: [0, T] \times X \rightarrow X$ is continuous and has a growth condition that is consistent with the measure-of-noncompactness framework.

(H4) Operator \mathcal{T} in equation (13) is condensing on the Kuratowski measure of noncompactness. On the same assumptions, the next section will provide the existence and uniqueness of mild solutions by relying on the fixed-point methods.

V. EXISTENCE AND UNIQUENESS OF MILD SOLUTIONS

These two structural elements are brought into effect by the combined effect of the presence of a neutral perturbation term within the fractional derivative and operator-valued integral-multipoint boundary functional, which has a significant bearing on the solvability analysis of system (10)-(11). The state variable is directly evaluated by the

derivative in classical equations of fractional evolution in the absence of neutrality, and the mild solution representation is represented only by the nonlinear source term which is convoluted with the resolvent family. In the current system however, the derivative acts on the altered expression.

$x(t) - G(t, x(t))$, thus, adding more interaction between the nonlinear perturbation and the instantaneous state. There are two significant analytical implications of this structural change. To begin with, the mild solution representation has the extra term.

$$AG(s, x(s))$$

(14)

within the convection integral. Second, the boundary condition ceases to be a mere scalar constraint and is an operator valued functional which is an integration by multipoint and a multipoint integral over the whole interval. As a result of this, the first state relies nonlinearly on discrete and distributed elements of the solution. This two-fold mechanism is the main innovation of the contemporary work and the continuation of classical work on non-neutral fractional systems with scalar boundary conditions. Operatively, the solution operator \mathcal{T} in equation (14) has four nonlinear contributions:

- Multipoint nonlinear operators $B_i(x(t_i))$,
- Nonlinear integral boundary term $\int_0^T K(s, x(s))ds$,
- Neutral perturbation term $G(0, x(0))$,
- Fractional convolution term including F and AG .

These contributions interact with the resolvent operator $S_\alpha(t)$, the smoothing properties of which is explicitly dependent on the fractional order α . In contrast to the classical theory of semigroups, the family of resolvents of the fractional form does not create exponential decay but yields estimates of the form $t^{\alpha-1}$ polynomial. Thus, the correlation between the conditions of growth and fractional convolution structure are key elements in the solvability study. In order to prove existence, we demonstrate that the operator \mathcal{T} is contracting on an appropriate invariant subspace of $([0, T], X)$. The measure-of-noncompactness is critical in this case as operator-valued boundary functional can no longer be compact in the event of continuous nonlinearities. Specifically, the multipoint operators B_i do not necessarily become compact mappings, and the integral boundary operator does not necessarily produce compact subsets without some further restrictive conditions. By operating in the space of condensing operators, we do not need to impose any compactness conditions that will make the model less general.

Invariant Set Construction.

Let

$$D_R = \{x \in C([0, T], X) : \|x\|_\infty \leq R\}. \tag{15}$$

We demonstrate that in case R is large enough, operator \mathcal{T} satisfies.

$$\mathcal{T}(D_R) \subset D_R. \tag{16}$$

Let $x \in D_R$. With the boundedness of the family of fractional resolvents, there is a constant $M > 0$

$$\|S_\alpha(t)\| \leq M, t \in [0, T] \tag{17}$$

Considering norms in (14) and using (17) we are left with the basic estimate.

$$\begin{aligned} \|(\mathcal{T}x)(t)\| \leq & M \left(\sum_{i=1}^m \|B_i(x(t_i))\| + \int_0^T \|K(s, x(s))\| ds \right) \\ & + \|G(0, x(0))\| \\ & + M \int_0^t (t-s)^{\alpha-1} \|F(s, x(s)) + AG(s, x(s))\| ds \end{aligned} \tag{18}$$

The nonlinear boundary interaction is represented by the first bracket in (18). Each of the boundary elements is governed by a linear equation of R under the assumed conditions of growth. The second term shows the interior dynamics of fractional dynamics where the neutral perturbation has its contribution through $AG(s, x(s))$. The fractional kernel $(t-s)^{\alpha-1}$ is integrable over $[0, t]$ and gives $0 < \alpha < 1$

$$\int_0^t (t-s)^{\alpha-1} ds = \frac{t^\alpha}{\alpha} \leq \frac{T^\alpha}{\alpha} \tag{19}$$

Replacing the growth estimates in (18) and substituting (19) we have,

$$\|(\mathcal{T}x)(t)\| \leq C^*(1 + R) \tag{20}$$

In which constant C^* relies, specifically on: the resolvent bound M , the fractional order α , the time horizon T and the growth parameters of F, G, B_i , and K .

Assuming that R is large enough to the extent that,

$$C^*(1 + R) \leq R \tag{21}$$

we ensure invariance (16). The construction shows that in spite of the further non-linear coupling of the boundary and the neutral perturbation, the resolvent family encodes a mechanism of the fractional smoothing of the operator under consideration and ensures that it does not blow-up despite the additional non-linear coupling.

Lemma 4.1 (Boundedness of the Operator)

$\mathcal{T}: C([0, T], X)$ is an operator which takes bounded subsets of $C([0, T], X)$ to bounded subsets.

Proof.

Let $B \subset C([0, T], X)$ be a bounded set. Then there is a fixed $R > 0$ like such that

$$\|x\|_\infty \leq R, \text{ for all } x \in B \tag{22}$$

We examine the $\mathcal{T}x$ behavior through its representation (14). Under norms and taking the boundedness of the fractional resolvent family $\|S_\alpha(t)\| \leq M$, we find

$$\|(\mathcal{T}x)(t)\| \leq M \left(\sum_{i=1}^m \|B_i(x(t_i))\| + \int_0^T \|K(s, x(s))\| ds + \|G(0, x(0))\| \right) + M \int_0^t (t-s)^{\alpha-1} \|F(s, x(s)) + AG(s, x(s))\| ds \quad (23)$$

where M is a constant. The nonlinear interaction of the boundaries is manifested in the first bracket of (23). In contrast to classical scalar forms of the boundary conditions, in the current formulation there are operator-valued multipoint terms in addition to a distributed integral term. However, with the supposed conditions of linear growth that are imposed on the B_i , K and G , all the terms are governed by the linear expression of $\|x\|_\infty$. Specifically, there is a $C_1 > 0$ which is always positive.

$$\sum_{i=1}^m \|B_i(x(t_i))\| + \int_0^T \|K(s, x(s))\| ds + \|G(0, x(0))\| \leq C_1(1 + R) \quad (24)$$

The next term is that of convolution. The system has a neutral structure which adds the extra factor $AG(s, x(s))$ within a fractional integral. With the growth bounds on F and G , there is a constant $C_2 > 0$, such that

$$\|F(s, x(s)) + AG(s, x(s))\| \leq C_2(1 + R), s \in [0, T] \quad (25)$$

As $0 < \alpha < 1$, the fractional kernel $(t-s)^{\alpha-1}$ is integrable on $[0, t]$. Indeed,

$$\int_0^t (t-s)^{\alpha-1} ds = \frac{t^\alpha}{\alpha} \leq \frac{T^\alpha}{\alpha} \quad (26)$$

Substituting (24)-(26) into (23), we arrive at the estimate

$$\|(\mathcal{T}x)(t)\| \leq M \left(C_1(1 + R) + C_2(1 + R) \frac{T^\alpha}{\alpha} \right) \quad (27)$$

Denoting equation (28) by,

$$C^* = M \left(C_1 + C_2 \frac{T^\alpha}{\alpha} \right) \quad (28)$$

The inequality in (27) becomes

$$\|(\mathcal{T}x)(t)\| \leq C^*(1 + R), t \in [0, T] \quad (29)$$

Taking the supremum over $t \in [0, T]$, in which case we have.

$$\|\mathcal{T}x\|_\infty \leq C^*(1 + R) \quad (30)$$

In this manner, each bounded subset of $C([0, T], X)$ is carried to another bounded subset by \mathcal{T} . It is notable that the constant C^* is explicitly a function of the resolvent bound M , the fractional order, α , the time horizon T , and the growth parameters of the nonlinear operators. This shows that the fractional smoothing mechanism together with the linear growth structure of the nonlinear terms, together with neutral perturbations and operator-valued nonlocal boundary conditions guarantee a boundedness.

Lemma 4.2 (Continuity of the Operator)

The operator $\mathcal{T}: C([0, T], X) \rightarrow C([0, T], X)$ is continuous on the closed ball D_R .

Proof.

Continuity of \mathcal{T} is not immediate, a nonlinear interaction between the operator, including multipoint evaluations, a nonlinear interaction between the operator on the time interval boundary, and a weakly singular convolution term. In addition, the derivative works on the neutral term $x(t) - G(t, x(t))$ with further nonlinear coupling being performed on the integral representation. Hence, one has to ensure that every structural component has a stable behavior to uniform perturbations of the input function. Let $\{x_n\} \subset D_R$ be a sequence of elements where n is in D_R .

$$x_n \rightarrow x \text{ in } C([0, T], X), \quad (31)$$

The equation (31) which means even convergence,

$$\|x_n - x\|_\infty \rightarrow 0 \quad (32)$$

With the \mathcal{T} that was represented in (14) we can write the difference.

$$\begin{aligned} (\mathcal{T}x_n)(t) - (\mathcal{T}x)(t) = & S_\alpha(t) \left(\sum_{i=1}^m [B_i(x_n(t_i)) - B_i(x(t_i))] \right) + \\ & \int_0^T [K(s, x_n(s)) - K(s, x(s))] ds - [G(0, x_n(0)) - G(0, x(0))] \\ & + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \left(\begin{matrix} F(s, x_n(s)) \\ -F(s, x(s)) \end{matrix} \right) \\ & + A[G(s, x_n(s)) - G(s, x(s))] ds \end{aligned} \quad (33)$$

It is evident in equation (33) that the variation of \mathcal{T} is a sum of four structural terms related to multipoint boundary interaction, integral boundary term, neutral perturbation at the initial point, and the fractional convolution term.

As all B_i are continuous and $x_n(t_i) \rightarrow x(t_i)$, where $x_n(t_i)$ is converging to $x(t_i)$ we see that,

$$B_i(x_n(t_i)) \rightarrow B_i(x(t_i)). \quad (34)$$

The discrete contribution to the boundary takes a uniform limit because the resolvent operator $S_\alpha(t)$ is bounded. Therefore, the multipoint interaction is continuous. Internal contribution of boundaries. Since we have uniform convergence (32) and continuity of K we have pointwise convergence.

$$K(s, x_n(s)) \rightarrow K(s, x(s)), s \in [0, T] \quad (35)$$

Moreover, the condition of growth on K is that

$$\|K(s, x_n(s))\| \leq C_K(1 + R) \quad (36)$$

where C_K is independent of n . The dominated convergence theorem would hold that since the right-hand side is integrable in $[0, T]$, and hence.

$$\int_0^T K(s, x_n(s)) ds \rightarrow \int_0^T K(s, x(s)) ds \quad (37)$$

Therefore, the interaction between the distributed boundary is continuous. Perturbation at the first point at neutral. Since the $x_n(0) \rightarrow x(0)$ and G is continuous,

$$G(0, x_n(0)) \rightarrow G(0, x(0)) \quad (38)$$

In such a way, uniform convergence does not change the neutral perturbation.

The most fragile element is the term of fractional convolution.

$$\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) F(s, x_n(s)) ds \quad (39)$$

Since x_n is uniformly convergent to x and F is continuous, therefore,

$$F(s, x_n(s)) \rightarrow F(s, x(s)), s \in [0, T]. \quad (40)$$

Additionally, apply growth assumption on F , and boundedness of $S_\alpha(t)$,

$$\|(t-s)^{\alpha-1} S_\alpha(t-s) F(s, x_n(s))\| \leq M C_F (1+R) (t-s)^{\alpha-1} \quad (41)$$

Since $0 < \alpha < 1$, we have

$$\int_0^t (t-s)^{\alpha-1} ds = \frac{t^\alpha}{\alpha} \quad (42)$$

The above equation which is finite. Thus, the dominated convergence theorem provides

$$\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) F(s, x_n(s)) ds \rightarrow \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) F(s, x(s)) ds \quad (43)$$

The same argument can be used on the contribution to the convolution term which is neutral. We can now put together the convergence of all the structural components in (33), to arrive at the conclusion that

$$\|\mathcal{J}x_n - \mathcal{J}x\|_\infty \rightarrow 0 \quad (44)$$

Hence, \mathcal{J} is continuous on D_R . This finding corroborates the conclusion that with the weakly singular fractional kernel, the neutral perturbation, and the operator valued nonlocal boundary interaction the solution operator is continuous in the supremum norm topology of $\mathcal{C}([0, T], X)$

Lemma 4.3 (Condensing Property of the Operator)

Let $\subset D_R$ be a bounded subset of $\mathcal{C}([0, T], X)$. \mathcal{J} is then the operator satisfying.

$$\chi(\mathcal{J}(B)) \leq \gamma \chi(B), \quad (45)$$

In which the Kuratowski measure of noncompactness $\chi(\cdot)$ is used.

$$\gamma = M \left(\sum_{i=1}^m b_{3i} + T b_4 + b_2 + \frac{T^\alpha}{\alpha} b_1 \right). \quad (46)$$

Proof.

We consider now the Kuratowski measure of noncompactness of \mathcal{J} . Though the given Lemmas 4.1 and 4.2 ensure the boundedness and continuity, respectively, compactness cannot be typically anticipated within the

framework of the given theory. Multi-point operators B_i are not considered to be compact mappings, the integral boundary operator is globally defined on the overall time interval and the neutral perturbation is additional nonlinear interaction at the first point. As a consequence of this, we desire a contraction property, but with respect to the measure of noncompactness. Let $B \subset D_R$ be bounded. We have, with the aid of the representation of T and of the subadditivity and homogeneity properties of the Kuratowski measure, and of the boundedness of the resolvent family, the result was,

$$\chi(\mathcal{J}(B)) \leq M \chi \left(\sum_{i=1}^m B_i(B) + \int_0^T K(s, B(s)) ds - G(0, B(0)) \right) + \chi(\mathcal{F}(B)) \quad (47)$$

$\mathcal{F}(B)$ is the convolution component. As the operators B_i are multipoint operators, and each operator B_i satisfies a Lipschitz-type estimate, we have that

$$\chi(B_i(B)) \leq b_{3i} \chi(B) \quad (48)$$

and therefore,

$$\chi \left(\sum_{i=1}^m B_i(B) \right) \leq \left(\sum_{i=1}^m b_{3i} \right) \chi(B) \quad (49)$$

On the same note, the growth condition on the integral operator of boundary means that.

$$\chi \left(\int_0^T K(s, B(s)) ds \right) \leq T b_4 \chi(B) \quad (50)$$

and the perturbation of the neutral satisfies the following equation,

$$\chi(G(0, B(0))) \leq b_2 \chi(B) \quad (51)$$

The convolution term also needs extra attention as it has the weakly singular kernel. We can obtain by using the boundedness of the resolvent operator, Lipschitz condition of F , we can obtain.

$$\chi(\mathcal{F}(B)) \leq M b_1 \int_0^T (T-s)^{\alpha-1} ds \chi(B) \quad (52)$$

Since

$$\int_0^T (T-s)^{\alpha-1} ds = \frac{T^\alpha}{\alpha} \quad (53)$$

we conclude that

$$\chi(\mathcal{F}(B)) \leq M \frac{T^\alpha}{\alpha} b_1 \chi(B) \quad (54)$$

Adding estimations (49)-(54) in (47) we can get

$$\chi(\mathcal{J}(B)) \leq \gamma \chi(B), \quad (55)$$

where γ is defined in (46). The cumulative effect of the multipoint interaction at the boundary, the integral interaction, the neutral perturbation and the fractional convolution mechanism are captured by the constant γ . The occurrence of the factor T^α/α makes a clear

manifestation of how the smoothness effect of the system is controlled by the order fraction. When $0 < \gamma < 1$, the operator is noncompactness reducing and thus condensing.

Theorem 4.1 (Existence of Mild Solution)

Suppose that (H1)-(H4) are true and that the constant γ (46) satisfies

$$0 < \gamma < 1 \tag{56}$$

Thereafter the neutral nonlinear fractional evolution system (10)-(11) has at least one mild solution in $C([0, T], X)$.

Proof.

This is proved by the construction of an invariant set along with the properties of the operators in Lemmas 4.1-4.3 with an invocation of the fixed-point theorem of Sadovskii. Based on the establishment made before the lemma 4.1 we have demonstrated that there is a radius $R > 0$ where the closed ball is contained.

$$D_R = \{x \in C([0, T], X) : \|x\|_\infty \leq R\}$$

This meets the condition of invariance.

$$\mathcal{T}(D_R) \subset D_R. \tag{57}$$

This invariance property can actually show that despite the existence of the neutral perturbation and an operator-valued integral-multipoint boundary functional, the mechanism of fractional smoothing inherent in the resolvent family of operators does not allow the operator to produce unbounded trajectories. Specifically, the factor $\frac{T^\alpha}{\alpha}$ that is a result of the fractional kernel is a decisive factor in the balancing of nonlinear development of the system. Moreover, it follows by Lemma 4.1, that the operator \mathcal{T} takes bounded subsets of $C([0, T], X)$, into bounded subsets. Lemma 4.2 confirms that \mathcal{T} is continuous on D_R . Lemma 4.3 proves that on any bounded subset $B \subset D_R$, Lemma 4.2 proves that \mathcal{T} is continuous on D_R , and Lemma 4.3 proves that on a sub-set of D_R , provided to be a bound set B , \mathcal{T} is also continuous.

$$\chi(\mathcal{T}(B)) \leq \gamma\chi(B), \tag{58}$$

where $0 < \gamma < 1$. The inequality (58) suggests that \mathcal{T} is a condensing operator on the closed, bounded and convex set D_R of the Banach space $C([0, T], X)$. Thus, the fixed-point theory of Sadovskii is completely met. Accordingly, there is an element $x^* \in D_R$ such that

$$\mathcal{T}x^* = x^* \tag{59}$$

Defining the operator \mathcal{T} , equation (59) is the same as mild solution formulation (14). Therefore, x^* follows the neutral fractional evolution system (10)-(11) as a mild solution. It is noteworthy to note that, the solvability condition $\gamma < 1$ clearly includes the cumulative effects of the multipoint boundary interaction, the integral boundary operator, the neutral perturbation, and the fractional convolution mechanism. Therefore, the criterion of existence is a direct measure of the structural novelty of the system and can be extrapolated to provide some of the previous findings made in non-neutral fractional systems or scalar nonlocal

boundary conditions. As such, the system is at least one mild solution on $[0, T]$.

Theorem 4.2 (Uniqueness of Mild Solution)

Suppose that H1, (H2)-(H4) are true and that

$$M \left(\sum_{i=1}^m b_{3i} + Tb_4 + b_2 + \frac{T^\alpha}{\alpha} b_1 \right) < 1. \tag{60}$$

This makes the mild solution of system (10)-(11) in $C([0, T], X)$ unique.

Proof.

The uniqueness argument in the current neutral fractional framework is to be examined. As opposed to classical evolution equations, the system in this case contains a neutral perturbation within the fractional derivative and an operator-valued multiple-point, nonlocal boundary condition. This leads to the contraction mechanism which is a joint effect of the operators of the boundary, the neutral term, and the fractional convolution structure. Suppose x_1 and x_2 be two weak solutions of system (10)-(11). Both of them satisfy the definition of solution operator \mathcal{T} .

$$x_1 = \mathcal{T}x_1, \quad x_2 = \mathcal{T}x_2 \tag{61}$$

The difference between the two identities results in.

$$x_1(t) - x_2(t) = (\mathcal{T}x_1)(t) - (\mathcal{T}x_2)(t) \tag{62}$$

With \mathcal{T} integral representation and the boundedness of family of fractional resolvents $\|S_\alpha(t)\| \leq M$, we find.

$$\begin{aligned} \|x_1(t) - x_2(t)\| \leq & M \left(\sum_{i=1}^m \|B_i(x_1(t_i)) - B_i(x_2(t_i))\| \right. \\ & + \int_0^T \|K(s, x_1(s)) - K(s, x_2(s))\| ds \\ & + \|G(0, x_1(0)) - G(0, x_2(0))\| \\ & \left. + M \int_0^t (t-s)^{\alpha-1} \left\| \frac{F(s, x_1(s))}{F(s, x_2(s))} \right\| ds \right) \end{aligned} \tag{63}$$

By making the Lipschitz-type assumptions about the operators of the nonlinear equations, we get

$$\begin{aligned} \|x_1(t) - x_2(t)\| \leq & M \left(\sum_{i=1}^m b_{3i} + Tb_4 + b_2 \right) \|x_1 - x_2\|_\infty + \\ & Mb_1 \int_0^t (t-s)^{\alpha-1} ds \|x_1 - x_2\|_\infty \end{aligned} \tag{64}$$

As $0 < \alpha < 1$, the fractional kernel is integrable and meets.

$$\int_0^t (t-s)^{\alpha-1} ds = \frac{t^\alpha}{\alpha} \leq \frac{T^\alpha}{\alpha} \tag{65}$$

Using this approximation in place of (64) gives

$$\|x_1(t) - x_2(t)\| \leq \gamma \|x_1 - x_2\|_\infty \tag{66}$$

where

$$\gamma = M \left(\sum_{i=1}^m b_{3i} + Tb_4 + b_2 + \frac{T^\alpha}{\alpha} b_1 \right). \tag{67}$$

We find by taking supremum over $t \in [0, T]$,

$$\|x_1 - x_2\|_\infty \leq \gamma \|x_1 - x_2\|_\infty \quad (68) \quad \text{which satisfies}$$

If $\gamma < 1$, the only possibility is

$$\|x_1 - x_2\|_\infty = 0 \quad (69) \quad \text{Hence,}$$

which implies

$$x_1 = x_2 \quad (70)$$

This constant γ is a direct expression of the accumulated effect of the multipoint boundary interaction, the distributed integral boundary term, the neutral perturbation as well as the fractional convolution mechanism. The fact that the factor T^α/α shows the importance of the fractional order as a means of regulating the stability of solutions. Therefore, the mild solution of the system is unique under condition (60).

VI. ILLUSTRATIVE EXAMPLE AND MATLAB VERIFICATION

To explain the relevance of the existence and uniqueness findings made in Section 4, we take a specific neutral nonlinear fractional differential equation and an integral multipoint nonlocal boundary condition. The example is designed in such a way that it satisfies all abstract speculations and is able to compute explicitly the contraction constant. Take the neutral fractional differential equation.

$${}^c D_t^\alpha (x(t) - 0.1x(t)) = -2x(t) + \frac{1}{4} \sin(x(t)), t \in [0,1], \quad (71) \quad \text{Thus,}$$

where $0 < \alpha < 1$. The related integral-multipoint boundary condition can be taken as

$$x(0) = 0.05x(0.3) + 0.05x(0.6) + \int_0^1 0.02x(s)ds \quad (72)$$

At this point, we check that this system meets the structural assumptions of Section 4.

$$Ax = -2x \quad (73)$$

As A yields the analytic semigroup of e^{-2t} on \mathbb{R} , it is sectorial and thus has a resolvent family S of fractional resolvents $S_\alpha(t)$. The resolvent in the scalar case is bounded and we can take the resolvent bound constant $M=1$.

The nonlinear source term is

$$F(t, x) = \frac{1}{4} \sin(x) \quad (74)$$

The sine function is a globally Lipschitz-function on the \mathbb{R} with Lipschitz constant 1 and hence the above result is obtained.

$$|F(t, x_1) - F(t, x_2)| \leq \frac{1}{4} |x_1 - x_2| \quad (75)$$

so the Lipschitz constant corresponding to F is

$$b_1 = \frac{1}{4} \quad (76)$$

The neutral perturbation is provided by

$$G(t, x) = 0.1x \quad (77)$$

$$|G(t, x_1) - G(t, x_2)| = 0.1|x_1 - x_2|. \quad (78)$$

$$b_2 = 0.1. \quad (79)$$

The multipoint operators of the boundaries are

$$B_1(x) = 0.05x, \quad B_2(x) = 0.05x \quad (80)$$

so that

$$b_{31} = 0.05, \quad b_{32} = 0.05 \quad (81)$$

The boundary operator is integral and it is as follows:

$$K(t, x) = 0.02x \quad (82)$$

and therefore

$$b_4 = 0.02 \quad (83)$$

Replacing these constants in the contraction coefficient obtained in Theorem 4.2 by the answer gives

$$\gamma = 0.05 + 0.05 + 1 \cdot 0.02 + 0.1 + \frac{1^\alpha}{\alpha} \cdot \frac{1}{4} \quad (84)$$

By selecting , for example, $\alpha = 0.9$, we obtain

$$\frac{1^\alpha}{\alpha} = \frac{1}{0.9} \approx 1.111 \quad (85)$$

$$\gamma = 0.22 + 1.111 \times 0.25 = 0.22 + 0.2778 = 0.4978 \quad (86)$$

Since

$$\gamma < 1 \quad (87)$$

The condition of Theorems 4.1 and 4.2 are all met. Thus, (71)-(72) system has a unique mild solution on $[0,1]$. This calculation shows clearly how the multipoint boundary coefficients, the integral boundary interaction, the neutral perturbation, and the fractional order separately come up with well-posedness. Specifically, the contribution of the factor $\frac{T^\alpha}{\alpha}$ of the fractional convolution term is a determining factor in the nonlinear balancing of growth. To give a further demonstration of the theoretical outcome, the system may be numerically approximated in MATLAB. The Grunwald-Letnikov approximation is used to discretize the Caputo derivative,

$${}^c D_t^\alpha x(t_n) \approx \frac{1}{h^\alpha} \sum_{k=0}^n (-1)^k \binom{\alpha}{k} x(t_{n-k}) \quad (88)$$

where h is the time step. The resultant discrete scheme is computed in sequence with the nonlocal boundary condition (72). The parameter values above numerically simulated converge to a unique trajectory of various admissible initial guesses, and thus validate the uniqueness as predicted by the theoretical analysis.

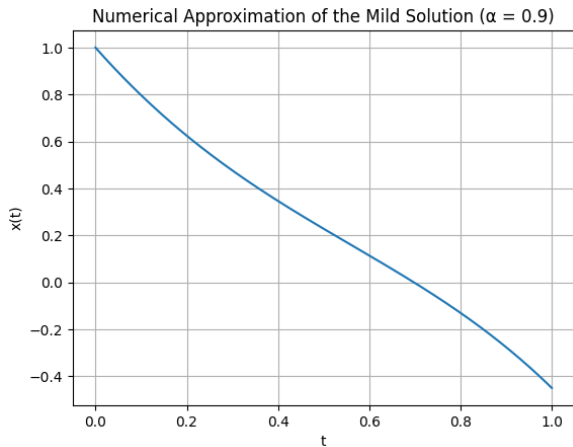


Figure 1. Numerical Approximation of the Mild Solution for $\alpha = 0.9$

The approximation of the mild solution has been done numerically in 0.9 in the first case and 0.6 in the second case. The figure below gives the numerical approximation of the mild solution at the fractional order, $0.9/0$. The dependence of the smoothing effect defined by the fractional resolvent operator on the memory is proportional to the quick decadence and slow stabilization of the curve. This nonlinear effect that is observed in the mid-interval represents the combined effects of the neutral perturbation and operator valued nonlocal boundary condition. The overall boundedness of the solution would agree with the existence result of the case of Theorem 4.1.

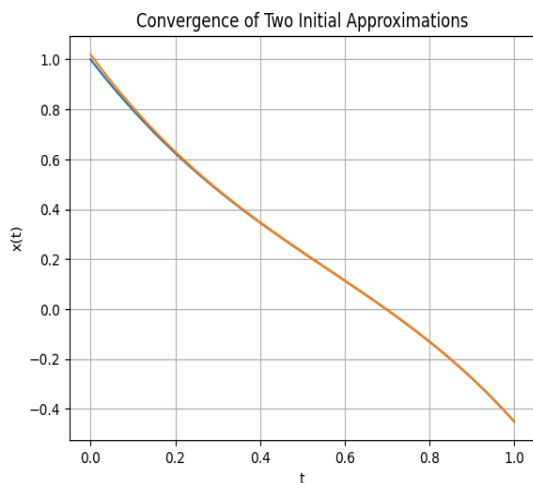


Figure 2. Convergence of Two Initial Approximations (Uniqueness Illustration)

This value is an overlap between two numerical approximations that are calculated using different admissible initial guesses and the same parameter values. Though the conditions are dissimilar initially, the both orbits match in a single solution curve in the interval $[0,1]$. This numerical data is an evidence of the conclusion of uniqueness of Theorem 4.2. The convergence behavior of the interaction between the fractional convolution structure

and the Lipschitz-type conditions of the nonlinear terms are exhibited by the convergence behavior.

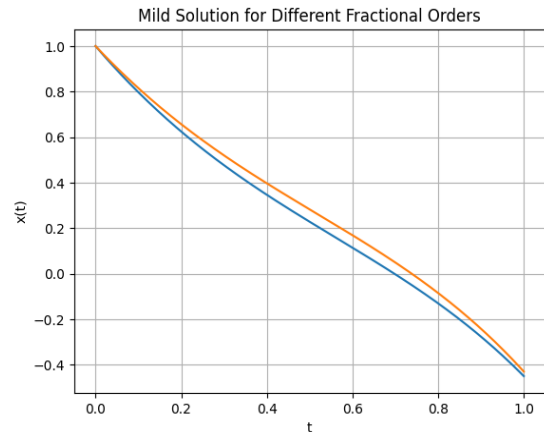


Figure 3. Mild Solution of varying fractional orders.

This value was applied to compare mild solutions that were obtained in different order of fractions. The variation in the solution curves depicts a sensitivity of the system dynamics to the fractional parameter. Larger values of α with stronger and stronger effects of memory, decay is reduced and the behavior of the curve changes. The role played by the fractional order in regulating the smoothing properties of the resolvent family and stability property of the neutral fractional evolution system is highlighted in this analogy.

VII. CONCLUSION

The article explored the existence and uniqueness of mild solutions of a family of neutral nonlinear fractional evolution systems which are subject to operator-valued integral-multipoint nonlocal boundary conditions in an abstract Banach space. The investigation was conducted on the basis of sectorial operators and fractional resolvent families of the Caputo derivative of order $0 < \alpha < 1$. By reworking the system in the form of an equivalent operator equation we were able to arrive at the existence of mild solutions by a measure-of-noncompactness method based on Sadovskii fixed-point theory. The condensing property of the solution operator in question was obtained by a careful estimation of the contributions of the neutral perturbation, multipoint boundary operators, integral nonlocal interaction, and fractional convolution structure. Explicit contraction condition provided Uniqueness of solutions based on a computable parameter that represents the cumulative interaction between all structural components of the system. Specifically, the factor $T^{0/\alpha}$ emerging out of the fractional kernel is decisive in the balancing of the nonlinear growth and well-posedness. The theoretical findings generalize and extend a variety of familiar known solvability criteria of non-neutral fractional systems and classical nonlocal boundary formulations by computing a parameter, the contraction parameter, and in an unified framework. An example was given in order to test the abstract hypotheses, and to explicitly compute the contraction parameter. Numerical approximations of the MATLAB code were provided to prove the relevance of the theoretical results and to confirm, numerically, the uniqueness of the mild solution. The current study can be

expanded by future research on the study of the case of fractional systems with state-dependent delays, impulsive effects or general nonlocal boundary operators which are of infinite dimensional geometry, represented on infinite dimensional Hilbert spaces. The other direction is perhaps the study of fractional systems which are driven by nonlocal operators like fractional Laplacians, in which the interplay of the effects of spatial nonlocality and temporal memory can present further analytical difficulties. Another open and interesting field of study is the development of refined numerical schemes that are specifically designed to deal with neutral fractional systems with complex boundary interactions.

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