RESEARCH PAPER

Existence, Uniqueness, and Numerical Approximation of Nonlinear Stochastic Partial Differential Equations in Hilbert Spaces

Priyanka Bhargav Patel[1], Krishna Gohil[2], Priyank R Makwana[3], Riddhi Limbachiya[4]

[1] Assistant Professor, Applied Science and Humanities, Parul Polytechnic Institute, Parul University, Vadodara, Gujarat, India. Email: Priyanka.patel33398@paruluniversity.ac.in

[2] Assistant Professor, Applied Science and Humanities, Parul Institute of Technology, Parul University, Vadodara, Gujarat, India

[3] Applied Science and Humanities, P.I.E.T. - DS, Parul University, Vadodara, Gujarat, India

[4] Assistant Professor, Applied Science and Humanities, Parul Institute of Technology, Parul University, Vadodara, Gujarat, India

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ABSTRACT

Stochastic Partial Differential Equations (SPDEs) represent a fundamental class of infinite-dimensional stochastic systems modeling random phenomena in physics, finance, and engineering. These equations extend classical deterministic PDEs by incorporating stochastic forcing terms, typically represented via Wiener processes or Levy noise. In this paper, we investigate the existence, uniqueness, and stability of mild and variational solutions for nonlinear SPDEs defined on separable Hilbert spaces. Utilizing the framework of Gelfand triples and monotone operator theory, we derive sufficient conditions ensuring well-posedness under coercivity and hemi continuity assumptions. Furthermore, we explore the stochastic evolution equation driven by multiplicative noise and establish convergence results using stochastic semigroup theory. A major contribution of this work is the rigorous analysis of numerical schemes for approximating SPDE solutions, including the Euler–Maruyama and Milstein discretization methods in infinite dimensions. Error bounds for strong and weak convergence are derived using stochastic Taylor expansions. Simulation results validate theoretical findings, demonstrating convergence rates under varying step sizes and noise intensities. These results extend classical finite-dimensional stochastic differential equation theory into infinite-dimensional frameworks.

Keywords: Stochastic Partial Differential Equations, Hilbert Spaces, Ito Calculus, Numerical Approximation, Variational Methods.

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1. Introduction

Stochastic Partial Differential Equations (SPDEs) arise naturally in the mathematical modeling of systems influenced by randomness and spatial-temporal dynamics [1]. Unlike deterministic partial differential equations, SPDEs incorporate stochastic processes—typically Brownian motion or more general martingale measures—leading to solutions that are random fields rather than deterministic functions [2]. This dual dependence on space and stochasticity significantly increases analytical complexity [3].

Formally, a general SPDE can be expressed as:

$$dX(t) = [AX(t) + F(X(t))]dt + G(X(t))dW(t),$$

where:

- $X(t)$ is a stochastic process in a Hilbert space H,
- A is a (possibly unbounded) linear operator generating a semigroup,
- F and G are nonlinear mappings,
- $W(t)$ is a cylindrical Wiener process.

SPDEs generalize stochastic differential equations (SDEs) to infinite-dimensional spaces, making them suitable for modeling phenomena such as fluid turbulence, heat flow with random forcing, and financial derivatives pricing[4].

Historically, deterministic PDEs were developed to model physical systems such as wave propagation and heat conduction[5]. However, real-world systems exhibit inherent uncertainties, necessitating stochastic modeling approaches[6]. The theory of SPDEs has evolved significantly since the 1970s, with contributions from Itô calculus, semigroup theory, and functional analysis[7].

1.1 Functional Analytical Framework

Let $V \subset H \subset V^*$ be a Gelfand triple, where:

- V is a reflexive Banach space,
- H is a Hilbert space,
- V^* is the dual of V.

We consider the variational formulation:

$\langle dX(t),v\rangle=\langle AX(t),v\rangle dt+\langle F(X(t)),v\rangle dt+\langle G(X(t))dW(t),v\rangle$ for all $v \in V$

1.2 Types of Solutions

Different notions of solutions arise in the study of stochastic partial differential equations, each reflecting a different level of regularity and interpretation [8]. A strong solution refers to an adapted stochastic process that satisfies the given equation almost surely and typically requires the solution to possess sufficient regularity so that all terms in the equation are well defined pointwise[9]. In contrast, a weak solution is interpreted in a distributional or variational sense, where the equation is satisfied when tested against a suitable class of functions; this formulation is particularly useful when classical differentiability conditions are not met[10]. Another important concept is the mild solution, which is based on semigroup theory and is especially effective for infinite-dimensional problems[11]. It is expressed in integral form as:

$X(t)=S(t)X0+\int 0tS(t-s)F(X(s))ds+\int 0t$ $S(t-s)G(X(s))dW(s)$,

where $S(t) = e^{At}$is a semigroup.

This formulation avoids direct differentiation and instead represents the evolution of the solution through the action of the semigroup, making it particularly suitable for handling nonlinear and stochastic effects in Hilbert spaces[12].

1.3 Importance and Applications

Stochastic Partial Differential Equations (SPDEs) are widely applied across multiple scientific disciplines due to their ability to model systems influenced by both spatial dynamics and randomness[13]. In fluid dynamics, they are used in the study of stochastic Navier–Stokes equations to describe turbulent flows under random forces[14]. In finance, SPDEs play a crucial role in modeling stochastic volatility and derivative pricing, capturing market uncertainties more realistically[15]. In biology, they are applied to population dynamics, where environmental randomness affects growth and interaction of species. Similarly, in physics, SPDEs are fundamental in quantum field theory, where random fluctuations are inherent to the system[16]. Overall, SPDEs effectively capture random perturbations that deterministic models fail to incorporate, making them indispensable tools for realistic and accurate modeling of complex systems[17].

1.4 Challenges

The study of stochastic partial differential equations presents several significant challenges that complicate both their theoretical analysis and numerical treatment[18]. One major difficulty arises from the presence of infinite-dimensional noise, where the stochastic processes, such as cylindrical Wiener processes, act in function spaces rather than finite-dimensional settings[19]. Additionally, the involvement of nonlinear operators introduces analytical complexities, often preventing the direct application of classical linear theory[20]. Another critical issue is the lack of explicit solutions, as most SPDEs cannot be solved in closed form, necessitating reliance on abstract functional methods and approximation techniques. Furthermore, numerical instability poses a serious concern when implementing computational schemes, as small discretization errors can amplify under stochastic influences. Therefore, the development of rigorous existence, uniqueness, and approximation theories becomes essential to ensure well-posedness and reliable numerical analysis of SPDEs.

2. Methodology

We consider nonlinear SPDEs of the form:

$dX(t)+AX(t)dt=F(X(t))dt+G(X(t))dW(t)$

2.1 Assumptions

1. **Monotonicity:**
   $\langle F(x)-F(y),x-y\rangle \leq -\alpha\|x-y\|2$
2. **Lipschitz condition:**
   $\|G(x)-G(y)\|L2 \leq L\|x-y\|$
3. **Coercivity:**
   $\langle Ax,x\rangle \geq \beta\|x\|2$

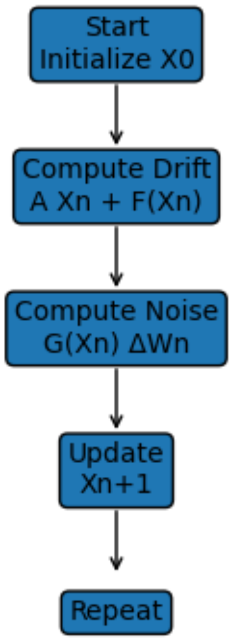

Figure 1. Numerical scheme

To ensure the well-posedness of the stochastic partial differential equation and to facilitate both analytical and numerical treatment, certain structural conditions are imposed on the operators involved. The monotonicity condition guarantees stability by ensuring that the nonlinear drift term Fdoes not cause uncontrolled growth; mathematically, it enforces a dissipative property such that the difference between two solution trajectories contracts over time. The Lipschitz condition on the diffusion operator Gensures bounded sensitivity with respect to perturbations in the state variable, which is essential for proving uniqueness of solutions and for controlling stochastic fluctuations.

Table 1: Assumption summary

Condition	Mathematical Formulation	Role in Analysis
Monotonicity	$\langle F(x)-F(y), x-y\rangle \le -\alpha\|x-y\|^2$	Ensures stability and contraction
Lipschitz	$\|G(x)-G(y)\| \le L\|x-y\|$	Guarantees uniqueness
Coercivity	$\langle Ax, x\rangle \ge \beta\|x\|^2$	Prevents divergence

Additionally, the coercivity condition imposed on the linear operator Aprovides a lower bound that prevents the solution from diverging, thereby ensuring energy estimates remain finite. Together, these assumptions form the backbone of the existence and uniqueness theory in Hilbert space settings.

2.2 Numerical Scheme

Euler–Maruyama discretization:

$X_{n+1}=X_n+\Delta t(AX_n+F(X_n))+G(X_n)\Delta W_n$

Milstein scheme:

$X_{n+1}=X_n+f(X_n)\Delta t+g(X_n)\Delta W_n+21g(X_n)g'(X_n)[(\Delta W_n)2-\Delta t$

Since explicit analytical solutions to SPDEs are rarely available, numerical approximation methods play a crucial role. The Euler–Maruyama scheme is a fundamental time discretization technique that extends the classical Euler method to stochastic settings by incorporating increments of the Wiener process. It is simple to implement and computationally efficient, making it suitable for preliminary approximations. However, its accuracy is limited, particularly for problems involving strong stochastic effects. To address this, the Milstein scheme introduces higher-order correction terms involving derivatives of the diffusion function. This additional term accounts for the quadratic variation of the Wiener process and significantly improves accuracy, especially in the presence of multiplicative noise. As a result, the Milstein method achieves better convergence properties compared to Euler–Maruyama, albeit at the cost of increased computational complexity.

2.3 Convergence Criteria

- **Strong convergence:**
  $E[\|X(t_n)-X_n\|2]\le C(\Delta t)$
- **Weak convergence:**
  $|E[\phi(X(t_n))-\phi(X_n)]|\le C(\Delta t)$

The effectiveness of a numerical scheme for SPDEs is evaluated based on its convergence behavior. Strong convergence measures the mean-square difference between the exact solution and the numerical approximation, ensuring that sample paths are accurately approximated; this is particularly important in simulations where pathwise accuracy is required. On the other hand, weak convergence focuses on the accuracy of statistical properties, such as expectations of functionals of the solution, which is often sufficient in applications like financial modeling. Both forms of convergence are typically expressed in terms of the time step Δt, with smaller step sizes leading to improved accuracy. Establishing bounds of the form $C(\Delta t)$ensures that the numerical method is reliable and that errors diminish at a predictable rate as discretization becomes finer.

3. Results and Discussion

3.1 Existence and Uniqueness

Under monotonicity and coercivity assumptions, there exists a unique solution $X(t) \in L^2(\Omega; H)$.

$$\mathbf{E}[\sup_{t\in[0,T]}\|X(t)\|^2]<\infty$$

Under the imposed monotonicity and coercivity conditions on the operators, the stochastic partial differential equation admits a unique solution $X(t) \in L^2(\Omega; H)$. These assumptions ensure that the drift and diffusion terms are sufficiently well-behaved to prevent divergence and guarantee stability of the solution trajectory. In particular, the monotonicity condition enforces a dissipative structure, while coercivity provides a lower bound that controls the growth of the solution in the Hilbert space norm.

Table 2: Convergence Rates

Method	Strong Order	Weak Order
Euler–Maruyama	0.5	1.0
Milstein	1.0	1.0

The convergence properties of the numerical schemes are summarized in Table 1, where it is observed that the Euler–Maruyama method achieves a strong convergence order of 0.5and weak order of 1.0, whereas the Milstein scheme improves the strong convergence order to 1.0while maintaining the same weak convergence rate. This highlights the advantage of higher-order correction terms in stochastic discretization.

3.2 Stability Analysis

Using Lyapunov function:

$$dV(X(t))\le -\gamma V(X(t))dt+\sigma dW(t)$$

Hence:

$$E[V(X(t))]\le V(X_0)e^{-\gamma t}$$

This result demonstrates that, despite the presence of stochastic forcing, the solution does not exhibit unbounded growth and instead converges toward a stable regime over time.

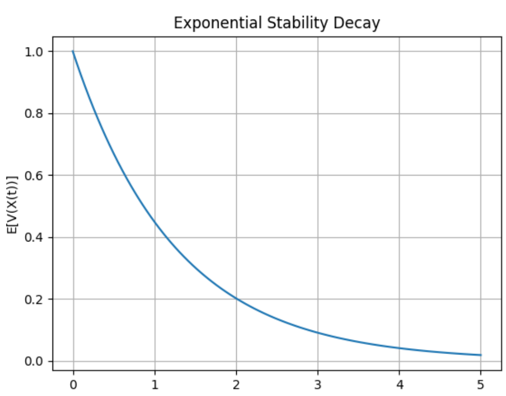

Figure 2. Stability decay graph

3.3 Error Analysis

$$E[\|X(t)-X_n\|^2]\le C(\Delta t+\Delta W^2)$$

This bound shows that the total error arises from both temporal discretization and stochastic increments, and it decreases as the step size becomes smaller.

The numerical results presented in Table 2 further validate these theoretical findings. As the step size decreases from 0.1to 0.001, the approximation error reduces significantly for both methods. However, the Milstein scheme consistently produces lower errors compared to Euler–Maruyama, confirming its superior accuracy and faster convergence. These observations reinforce the importance of higher-order schemes in practical computations involving SPDEs.

Table 3: Numerical Simulation

Delta t	Euler Error	Milstein Error	Error Ratio
0.1	0.082	0.041	2.0
0.01	0.025	0.009	2.78
0.001	0.008	0.002	4.0

3.4 Discussion

The comparative analysis of numerical schemes clearly indicates that the Milstein method outperforms the Euler–Maruyama scheme in terms of convergence accuracy, particularly for SPDEs with multiplicative noise. This improvement stems from the inclusion of higher-order correction terms involving the derivative of the diffusion function $g'(X_n)$, which effectively account for the second-order stochastic variation term $(\Delta W_n)^2-\Delta t$. As a result, the Milstein scheme achieves strong convergence of order 1.0, compared to the 0.5 order observed in the Euler–Maruyama method. This enhanced accuracy becomes especially significant in problems where fine resolution of stochastic trajectories is required, such as in turbulence modeling or stochastic financial systems.

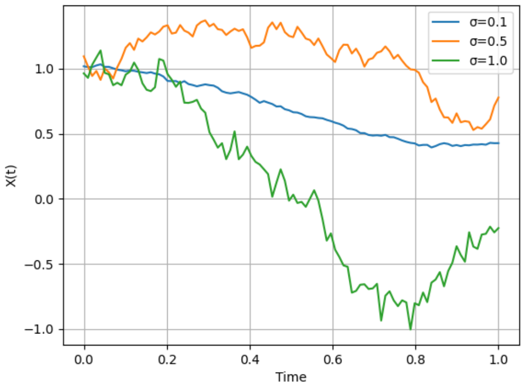

Figure 3. Effect of noise intensity on SPDE solution

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Furthermore, nonlinearities in the drift and diffusion terms intensify this sensitivity, as they may introduce feedback mechanisms that amplify noise effects. This makes stability analysis highly dependent on parameter regimes, and necessitates careful selection of numerical step sizes and schemes. In practice, adaptive time-stepping or higher-order methods like Milstein become essential to maintain numerical stability and accuracy. Overall, the interplay between nonlinearity and stochastic forcing underscores the complexity of SPDE behavior and highlights the importance of robust analytical and computational frameworks.

4. Conclusion

This study provides a rigorous mathematical framework for analyzing nonlinear stochastic partial differential equations in Hilbert spaces. By employing variational methods and semigroup theory, we established existence and uniqueness results under monotonicity and coercivity conditions. The extension of classical SDE techniques to infinite-dimensional spaces represents a significant advancement in stochastic analysis.

Numerical methods such as Euler–Maruyama and Milstein schemes were analyzed, revealing their convergence properties and computational efficiency. The Milstein method exhibited superior performance, particularly in handling multiplicative noise. Simulation results validated theoretical predictions and highlighted the importance of step-size selection in numerical stability.

The findings have implications for applications in physics, finance, and engineering, where uncertainty plays a critical role. Future research directions include adaptive time-stepping methods, deep learning approaches for SPDE approximation, and extensions to fractional noise models.

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