GENERALISED FRACTIONAL DECOMPOSITION-SERIES METHOD FOR MULTI-ORDER COUPLED NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH PHARMACOKINETIC APPLICATIONS

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Abstract: Fractional differential equations have become a fundamental modelling tool for complex systems exhibiting memory, hereditary dynamics, and anomalous transport. The present paper introduces the ***Generalised Fractional Decomposition-Series Method (GFDSM)***, a rigorous continuation and substantial extension of three preceding works by the same authors namely a Novel Analytical Approach (Paper I), the Hybrid Analytical Framework (HAF, Paper II), and a Unified Analytical Framework (Paper III) all directed at nonlinear fractional differential equations (NFDEs) in engineering contexts. The GFDSM advances the theoretical edifice in three decisive directions: (i) a *Multi-Order Coupled Existence Theorem* that generalises Banach contraction theory to systems of NFDEs with simultaneous Caputo derivatives of distinct rational orders and fully general coupling matrices; (ii) a *Fractional Spectral Convergence Theorem* that derives sharp, time-dependent error bounds incorporating the Mittag-Leffler growth factor; and (iii) a *Pharmacokinetic Fractional Transport Theorem (PFTT)* that bridges the abstract operator theory with concrete drug-delivery applications by modelling anomalous drug absorption and elimination through multi-compartment fractional systems. Four additional theorems are established and rigorously proved, covering adaptive residual correction, spectral stability, generalised error propagation, and fractional Lyapunov stability for coupled nonlinear systems. The GFDSM is applied to five engineering and pharmacokinetic models: a coupled thermo-viscoelastic oscillator, a multi-order SIR epidemiological system, a fractional nonlinear Schrödinger equation, a coupled electrochemical circuit, and a two-compartment anomalous pharmacokinetic model with nonlinear protein binding. Comparative analysis against classical ADM, HPM, HAF, and numerical Caputo solvers demonstrates that GFDSM achieves exponential residual decay and errors below 10^{-10} with ten series terms across all fractional orders $\alpha \in (0, 1]$, confirming the superior accuracy and generality of the proposed framework.

Keywords: Fractional Differential Equations; Caputo Derivative; Multi-Order Coupled Systems; Generalised Decomposition-Series Method; Mittag-Leffler Stability; Pharmacokinetic Drug Delivery; Anomalous Diffusion; Convergence Analysis; Nonlinear Dynamics; Hybrid Analytical Method.

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. INTRODUCTION

Fractional calculus has emerged as one of the most powerful and versatile tools for the mathematical modelling of physical, biological, and engineering phenomena that involve memory, hereditary properties, and nonlocal spatial interactions. Unlike integer-order derivatives, the Caputo fractional derivative encodes the entire history of a dynamical system into its present rate of change, making it ideally suited for modelling viscoelastic deformation, anomalous diffusion, dielectric polarisation, electrochemical processes, and pharmacokinetic drug transport [1–4]. Over the past two decades, considerable research effort has been directed at developing reliable analytical and numerical methods for nonlinear fractional differential equations (NFDEs), motivated by the recognition that closed-form solutions are rarely attainable for such systems and that classical perturbation or numerical methods often suffer from convergence limitations, accumulated truncation errors, and instability when applied to strongly nonlinear fractional operators [5–8].

The authors' preceding sequence of papers established a progressively deepening analytical foundation for NFDEs. Paper I [24] introduced a novel framework built upon the *Fractional Nonlinear Superposition Theorem (FNST)*, the *Fractional Stability Criterion*

(FSC), the *Fractional Convergence Theorem (FCT)*, and the *Generalized Fractional Transformation Theorem (GFTT)*, validated on viscoelastic, thermal, circuit, and biological models. Paper II [25] proposed the *Hybrid Analytical Framework (HAF)*, synergistically combining the Homotopy Perturbation Method (HPM) and the Adomian Decomposition Method (ADM) with a fractional residual correction mechanism, and proved the *Hybrid Analytical Convergence Theorem* guaranteeing uniform convergence under Lipschitz continuity conditions. Paper III [26] developed a *Unified Analytical Framework (UAF)* using integral reformulation and Banach fixed-point theory, applying it to fractional oscillators, heat conduction, and electrical circuits with memory elements.

Despite these advances, three substantive limitations remain unaddressed in the existing series. First, all theoretical results pertain to single-order Caputo systems; the most practically important engineering and pharmacokinetic models involve coupled systems with simultaneous Caputo derivatives of distinct rational orders, and no general contraction theory for such systems has been established within this research programme. Second, error bounds derived in Papers I–III are static (independent of time), failing to capture the Mittag-Leffler growth of the fractional integral operator that becomes significant for long-time horizons. Third, and most importantly for the scope of the present journal, no connection has been drawn between the developed analytical framework and drug-delivery applications, despite the well-documented role of fractional-order pharmacokinetics in modelling anomalous drug absorption, nonlinear protein binding, and memory-dependent drug release from sustained-delivery matrices [9–12].

The present paper addresses all three limitations through the ***Generalised Fractional Decomposition-Series Method (GFDSM)***. The GFDSM retains and extends the structural core of HAF namely the multi-component Adomian decomposition with adaptive residual correction while introducing: (i) a fully coupled multi-order existence and uniqueness theory governed by the spectral radius of the Lipschitz coupling matrix; (ii) sharp, time-dependent error bounds incorporating the Mittag-Leffler factor; (iii) a Pharmacokinetic Fractional Transport Theorem (PFTT) linking the operator framework to drug-compartment modelling; and (iv) spectral stability results for spatially distributed fractional systems. The paper is structured as follows. Section 2 establishes extended mathematical preliminaries. Section 3 develops the GFDSM algorithm in full generality. Section 4 states and proves the six new theorems. Section 5 applies the GFDSM to five representative systems. Section 6 provides comparative numerical analysis. Section 7 concludes with directions for future research.

2. MATHEMATICAL PRELIMINARIES

This section establishes the notation, operator definitions, and foundational lemmas required for the GFDSM. All definitions are formulated for the multi-order coupled setting that generalises the single-order framework of Papers I–III.

2.1 Banach Space and Norm

Let $J = [0, T]$ for fixed $T > 0$ and let $n \geq 1$ denote the number of coupled system components. The Banach space $C(J, \mathbb{R}^n)$ of continuous vector-valued functions is equipped with the supremum norm $\|U\|_\infty = \max_i \sup_{t\in J} |u_i(t)|$. For the spatial setting, $L^2\_w(0,L)$ denotes the weighted Hilbert space with inner product $\langle f,g\rangle = \int_0^L f(x)g(x)w(x)dx$, where $w(x) > 0$ is a weight function.

2.2 Caputo Fractional Derivative

Definition 2.1 (Caputo Fractional Derivative). For $u(t) \in C^n(J)$ and $n\text{-}1 < \alpha < n$, $n \in \mathbb{N}$, the Caputo derivative of order α is:

$${}^c D\_t\text{^}\alpha\, u(t) = \Gamma(n\text{-}\alpha)^{-1} \int_0^t (t\text{-}\tau)^{n-\alpha-1} u^n(\tau)\, d\tau, \quad n\text{-}1 < \alpha < n.$$

This definition permits integer-order initial conditions $u^k(0) = c^k$, $k = 0,1,\ldots,n-1$, making it preferable to the Riemann-Liouville operator in all engineering and pharmacokinetic applications in this paper. The Caputo operator satisfies linearity and the composition identity $I\_t\text{^}\alpha\ {}^c D\_t\text{^}\alpha\, u(t) = u(t) - u(0)$ for $0 < \alpha < 1$.

2.3 Fractional Integral Operator and Boundedness Lemma

Definition 2.2 (Fractional Integral). The Riemann-Liouville fractional integral of order $\alpha > 0$ is:

$$I\_t\text{^}\alpha\, u(t) = \Gamma(\alpha)^{-1} \int_0^t (t\text{-}\tau)\text{^}\alpha^{-1}\, u(\tau)\, d\tau.$$

Lemma 2.1 (Boundedness of I_t^α). For $u \in C(J)$ and $\alpha > 0$:

$$\|I\_t\text{^}\alpha\, u\|_\infty \leq T\text{^}\alpha / \Gamma(\alpha+1) \cdot \|u\|_\infty.$$

Proof: follows by direct estimation of the convolution kernel (see Paper II, Lemma 1 for the detailed derivation).

2.4 Fractional Grönwall Inequality

Lemma 2.2 (Fractional Grönwall Inequality [13]). If $v(t) \geq 0$ satisfies $v(t) \leq c + k \int_0^t (t\text{-}\tau)\text{^}\alpha^{-1}\, v(\tau)\, d\tau$ for constants $c, k \geq 0$, then:

$$v(t) \leq c \cdot E\_\alpha(k\, \Gamma(\alpha)\, t\text{^}\alpha),$$

where $E\_\alpha(\cdot)$ is the one-parameter Mittag-Leffler function.

2.5 Mittag-Leffler Functions

The one-parameter and two-parameter Mittag-Leffler functions are defined by:

$$E\_\alpha(z) = \Sigma\_\{k=0\}\text{^}\infty\ z^k / \Gamma(\alpha k+1), \qquad E\_\{\alpha,\beta\}(z) = \Sigma\_\{k=0\}\text{^}\infty\ z^k / \Gamma(\alpha k+\beta).$$

The decay bound $|E\_\alpha(\text{-}\lambda t\text{^}\alpha)| \leq C/(1+\lambda t\text{^}\alpha)$ for $\lambda > 0$ governs the algebraic relaxation characteristic of fractional systems. This bound will be used throughout Sections 4 and 5.

3. THE GENERALISED FRACTIONAL DECOMPOSITION-SERIES METHOD (GFDSM)

3.1 Problem Formulation

Consider the coupled multi-order nonlinear fractional system:

$^{c}D\_t^{\alpha_i}\, u_i(t) = F_i(t, u_1(t), u_2(t), \ldots, u_n(t)), \quad i = 1,\ldots,n, \; t \in J,$

subject to $u_i(0) = u_{i0} \in \mathbb{R}$. Here $0 < \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n \leq 1$ are distinct rational fractional orders, and $F_i : J \times \mathbb{R}^n \to \mathbb{R}$ are jointly continuous nonlinear functions satisfying the Lipschitz coupling condition (see Theorem 1 below). Applying the component-wise fractional integral operator $\hat{0}^\alpha$ yields the equivalent vector Volterra integral equation:

$U(t) = U_0 + \hat{0}^\alpha F(t, U(t)),$

where $U = (u_1,\ldots,u_n)^T$ and $F = (F_1,\ldots,F_n)^T$.

3.2 Multi-Component Series Decomposition

Write each component as the convergent series $u_i(t) = \Sigma\_\{m=0\}^\infty\, u_{i,m}(t)$. Expand each nonlinear term $F_i(t, U)$ using the generalised multi-variable Adomian polynomials:

$^{e}C_{i,m} = (1/m!)\, [d^m/d\lambda^m\, F_i(t, \Sigma\_\{k=0\}^m\, \lambda^k\, U\_k)]^T\_\{\lambda=0\},\; m = 0,1,2,\ldots$

The GFDSM recursive scheme for each component is:

$u_{i,0}(t) = u_{i0}, \quad u_{i,m+1}(t) = I\_t^{\alpha_i}\, [^{e}C_{i,m}(t)], \quad m \geq 0.$

3.3 Adaptive Spectral Residual Correction

After computing the M-term truncation $U\_M(t) = \Sigma\_\{m=0\}^T\, U\_m(t)$, define the coupled residual:

$R\_M(t) = {}^{c}D\_t^\alpha\, U\_M(t) - F(t, U\_M(t)).$

The spectral correction weight λ_i^* for component i minimises the Sobolev norm of the residual:

$\lambda_i^* = \langle R\_\{M,i\},\; I\_t^{\alpha_i}\; R\_\{M,i\}\rangle\_\{L^2\} \;/\; \|I\_t^{\alpha_i}\, R\_\{M,i\}\|^2\_\{L^2\}.$

The corrected approximation is then:

$U\_\{M+1\}(t) = U\_M(t) - \Lambda^*\; \hat{0}^\alpha\; R\_M(t), \quad \Lambda^* = diag(\lambda_1^*,\ldots,\lambda_n^*).$

This adaptive strategy, extending the static correction parameter of HAF, exploits the spectral geometry of the residual in H^α-space and achieves a reduction factor equal to the Rayleigh quotient of the fractional integral operator at each iteration.

3.4 GFDSM Step-by-Step Algorithm

Step 1. Write the NFDE system in vector integral form $U = U_0 + \hat{0}^\alpha F(t,U)$.

Step 2. Initialise $U_0 = (u_{10},\ldots,u_{n0})^T$. Compute multi-variable Adomian polynomials $^{e}C_{i},m$.

Step 3. Recursively compute GFDSM series components U_m until $\|R\_M\|\_\{H^\alpha\} \leq \varepsilon$ or $m = M\_\{max\}$.

Step 4. Compute optimal spectral weights λ_i^* by solving the component-wise minimisation.

Step 5. Apply adaptive residual correction: $U\_\{M+1\} = U\_M - \Lambda^*\; \hat{0}^\alpha\; R\_M$. If $\|R\_\{M+1\}\| \leq \varepsilon$, accept $U\_\{M+1\}$; otherwise return to Step 3.

4. MAIN RESULTS: THEOREMS AND PROOFS

4.1 Theorem 1 — Multi-Order Contraction Theorem (MOCT)

Theorem 1 (Multi-Order Contraction Theorem — MOCT). Let $F : J \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and satisfy the coupled Lipschitz condition:

$|F_i(t,U) - F_i(t,V)| \leq \Sigma_j\, L_{ij}\, |u_j - v_j|, \quad \forall t \in J, \quad U,V \in \mathbb{R}^n,$

where $L = [L_{ij}]$ is the Lipschitz coupling matrix. Define $\hat{2}_{ij} = L_{ij}\; T^{\alpha_i} \,/\, \Gamma(\alpha_i+1)$ and let $\rho(\hat{2})$ denote its spectral radius. If $\rho(\hat{2}) < 1$, then the coupled integral operator $\hat{B}$ is a contraction on $C(J, \mathbb{R}^n)$ and the system has a unique solution $U^* \in C(J, \mathbb{R}^n)$.

Proof. For any $U, V \in C(J, \mathbb{R}^n)$, estimate component i:

$|(\hat{B}U)_i(t) - (\hat{B}V)_i(t)| \leq I\_t^{\alpha_i}\, |F_i(t,U) - F_i(t,V)| \leq \Sigma_j\, L_{ij}\, I\_t^{\alpha_i}\, |u_j - v_j|.$

Taking the supremum over J and applying Lemma 2.1: $\|(\hat{B}U)_i - (\hat{B}V)_i\|_\infty \leq \Sigma_j\, \hat{2}_{ij}\, \|u_j - v_j\|_\infty.$

Assembling in vector form: $\|\hat{B}U - \hat{B}V\| \leq \rho(\hat{2})\, \|U - V\|$. Since $\rho(\hat{2}) < 1$, $\hat{B}$ is a contraction. The Banach Fixed-Point Theorem guarantees existence and uniqueness of $U^* \in C(J, \mathbb{R}^n)$ with $\hat{B}U^* = U^*$. When $n = 1$ and $\alpha_1 = \alpha$, Theorem 1 reduces to the classical uniqueness results of Papers I–III, confirming backward compatibility.

4.2 Theorem 2 — Fractional Spectral Convergence Theorem (FSCT)

Theorem 2 (Fractional Spectral Convergence Theorem — FSCT). Under the conditions of Theorem 1, let $\{U\_m\}\_\{m=0\}^\infty$ be the GFDSM sequence. Then:

(i) The sequence converges uniformly to U^* in $C(J, \mathbb{R}^n)$.

(ii) The total truncation error satisfies the time-dependent bound:

$\|U^* - U\_M\|_\infty \leq [\rho(\hat{2})^{T+1} \,/\, (1 - \rho(\hat{2}))] \cdot \|U_0\|_\infty \cdot E\_\{\alpha\_max\}(\rho(\hat{2})\; T^{\alpha\_max}),$

where $\alpha\_\{max\} = max_i\; \alpha_i$.

(iii) After each adaptive spectral correction step, the residual satisfies:

$\|R\_\{M+1\}\|\_\{H^\alpha\} \leq (1 - \rho_i^2)\; \|R\_M\|\_\{H^\alpha\},$

where ρ_i is the Rayleigh quotient of $I\_t^{\alpha_i}$ at $R\_\{M,i\}$.

Proof (Sketch). Part (i) is immediate from the contraction property of $\hat{B}$ (Theorem 1) and the standard Banach iteration $U\_\{m+1\} = \hat{B}U\_m$. For Part (ii), the bound $\|U^* - U\_M\| \leq \rho(\hat{2})^T/(1-\rho(\hat{2}))\; \|U_1 - U_0\|$ follows from the geometric series estimate; applying the fractional Grönwall inequality (Lemma 2.2) to bound $\|U_1 - U_0\| \leq \|U_0\|\; E\_\{\alpha\_\{max\}\}(\rho(\hat{2})T^{\alpha\_\{max\}})$ yields Part (ii). For Part (iii), the correction $U\_\{M+1\} = U\_M - \Lambda^*\; \hat{0}^\alpha\; R\_M$ is a steepest-descent step in H^α; computing $\|R\_\{M+1\}\|^2\_\{H^\alpha\}$ and optimising over λ_i^* yields the Rayleigh quotient reduction factor.

4.3 Theorem 3 — Pharmacokinetic Fractional Transport Theorem (PFTT)

Theorem 3 (Pharmacokinetic Fractional Transport Theorem — PFTT). Consider the two-compartment fractional pharmacokinetic model:

$^{c}D\_t^\alpha\; C\_p(t) = k\_a\; C\_d(t) - k\_e\; C\_p(t) - \mu\; C\_p^q(t), \quad 0 < \alpha \leq 1,$

$^{c}D\_t^\beta\; C\_d(t) = -k\_a\; C\_d(t), \quad 0 < \beta \leq 1,$

where $C\_p(t)$ and $C\_d(t)$ are plasma and depot drug concentrations, $k\_a, k\_e > 0$ are fractional absorption and elimination rate constants, $\mu \geq 0$ is the nonlinear protein-binding coefficient, and $q > 1$ governs the binding nonlinearity. Then:

(i) The system satisfies the conditions of Theorem 1 with coupling matrix $L\_\{11\} = k\_e + q\mu$

$C_{p,max}^{q-1}$, $L_{12} = k_a$, $L_{21} = 0$, $L_{22} = k_a$, and a unique solution exists on $J = [0, T]$ whenever $\rho(\hat{}2) < 1$.

(ii) The GFDSM series solution is:

$C_d(t) = C_{d0}\, E_\beta(-k_a\, t^\beta)$,

$C_p(t) = \Sigma_{m=0}^\infty A_m(t), \quad A_0 = C_{p0}$,

$A_{m+1}(t) = I_t^\alpha[k_a\, C_{d,m}(t) - k_e\, A_m(t) - \mu\, P_m(t)]$,

where $C_{d,m}(t) = C_{d0}\, E_\beta(-k_a\, t^\beta) \cdot (m^t$ *truncation*$)$ and P_m are the Adomian polynomials for the nonlinear binding term C_p^q.

(iii) The plasma concentration satisfies the asymptotic Mittag-Leffler bound:

$|C_p(t)| \le C_{p0}\, E_\alpha(-k_e\, t^\alpha) + k_a\, C_{d0}\, t^\alpha\, E_{\alpha,\alpha+1}(-k_e\, t^\alpha), \quad t \in J.$

Proof. Part (i): the Lipschitz condition on $F_1 = k_a\, C_d - k_e\, C_p - \mu\, C_p^q$ follows from the mean-value theorem applied to C_p^q on a bounded domain, giving $L_{11} = k_e + q\mu\|C_p\|_{\infty}^{q-1}$. Since $F_2 = -k_a\, C_d$ is linear, $L_{21} = 0$, $L_{22} = k_a$, and the spectral radius condition $\rho(\hat{}2) < 1$ reduces to $k_a\, T^\beta/\Gamma(\beta+1) + k_e\, T^\alpha/\Gamma(\alpha+1) < 1$, which holds for sufficiently small T or sufficiently small rate constants.

Part (ii): The depot equation is a linear one-component Caputo equation with solution $C_d(t) = C_{d0}\, E_\beta(-k_a\, t^\beta)$ (standard result [13]). Substituting into the plasma equation and applying the GFDSM recursion of Section 3.2 gives the stated formulae.

Part (iii): Applying the fractional variation-of-parameters formula to the linearised plasma equation ($\mu = 0$) and using the integral representation of $E_{\alpha,\alpha+1}$ yields the stated bound. For $\mu > 0$, the Adomian correction terms contribute higher-order modifications that do not alter the leading-order Mittag-Leffler behaviour.

4.4 Theorem 4 — Fractional Lyapunov Stability Theorem (FLST)

Theorem 4 (Fractional Lyapunov Stability Theorem — FLST). Consider the coupled NFDE system of Section 3.1. Suppose there exists a Lyapunov functional $V(t, U) : J \times \mathbb{R}^n \to [0,\infty)$ satisfying: (a) $V(t,0) = 0$ and $c_1\|U\|^2 \le V(t,U) \le c_2\|U\|^2$ for positive constants c_1, c_2; and (b) ${}^c D_t^{\alpha_{max}} V(t,U(t)) \le -\eta\|U\|^2$ for some $\eta > 0$. Then the equilibrium $U = 0$ is Mittag-Leffler stable:

$\|U(t)\| \le (c_2/c_1)^{12}\, \|U_0\|\, [E_{\alpha_{max}}(-(\eta/c_2)\, t^{\alpha_{max}})]^{12}$.

Proof. Integrating the fractional Lyapunov inequality (b) using the fractional integral $I_t^{\alpha_{max}}$ and applying the Gronwall lemma (Lemma 2.2) to $V(t, U(t))$ yields $V(t,U) \le V(0,U_0)\, E_{\alpha_{max}}(-(\eta/c_2)t^{\alpha_{max}})$. Invoking (a) for the lower and upper bounds then gives the stated norm estimate.

4.5 Theorem 5 — Generalised Error Propagation Theorem (GEPT)

Theorem 5 (Generalised Error Propagation Theorem — GEPT). Under Theorem 1 conditions, the GFDSM truncation error at step M satisfies:

$\|U^* - U_M\|_\infty \le [\rho(\hat{}2)^{\wedge T} / (1-\rho(\hat{}2))] \cdot \|U_0\|_\infty \cdot E_{\alpha_{max}}(\rho(\hat{}2)\, T^{\alpha_{max}}), \quad M \ge 1.$

The factor $E_{\alpha_{max}}(\rho(\hat{}2)\, T^{\alpha_{max}})$ captures the memory-induced error growth absent from the static bounds in Papers I–III.

Proof. By contraction, $\|U^* - U_M\| \le \rho(\hat{}2)^{\wedge T}/(1-\rho(\hat{}2))\, \|U^* - U_0\|$. Bounding $\|U^*\|$ via the Grönwall inequality applied to $U^* = U_0 + \hat{}0^\alpha\, F(t,U^*)$ gives $\|U^*\| \le \|U_0\|\, E_{\alpha_{max}}(\rho(\hat{}2)\, T^{\alpha_{max}})$; the stated bound follows.

4.6 Theorem 6 — Fractional Anomalous Diffusion-Transport Theorem (FADTT)

Theorem 6 (Fractional Anomalous Diffusion-Transport Theorem — FADTT). Let $C(x,t)$ satisfy the nonlinear time-fractional anomalous diffusion-reaction equation modelling drug transport through a heterogeneous biological membrane:

${}^c D_t^\alpha\, C(x,t) = D_\alpha\, \partial^2 C/\partial x^2 - k_r\, C(x,t) + \gamma\, C^p(x,t) + s(x,t), \quad 0<\alpha<1, \quad x \in (0,L),$

with initial condition $C(x,0) = C_0(x)$ and homogeneous boundary conditions $C(0,t) = C(L,t) = 0$. Here $D_\alpha > 0$ is the anomalous diffusivity, $k_r > 0$ is the first-order clearance rate, $\gamma \ge 0$ is the nonlinear metabolic reaction coefficient, and $s(x,t)$ is an external source. Then:

(i) A unique solution $C \in C(J, L^2(0,L))$ exists provided $\gamma\, p\, C_{max}^{p-1}\, T^\alpha / \Gamma(\alpha+1) < 1$.

(ii) The GFDSM spectral expansion yields:

$C(x,t) = \Sigma_{n=1}^\infty\, [\hat{C}_n(0)\, E_\alpha(-\Lambda_n\, t^\alpha) + \int_0^t (t-s)^{\alpha-1}\, E_{\alpha,\alpha}(-\Lambda_n(t-s)^\alpha)\hat{G}_n(s)ds]\, \phi_n(x),$

where $\Lambda_n = D_\alpha\lambda_n + k_r$, $\lambda_n = n^2\pi^2/L^2$ are the Laplacian eigenvalues, $\phi_n(x) = \sqrt{(2/L)}\, \sin(n\pi x/L)$ are the corresponding eigenfunctions, and $\hat{G}_n(s) = \langle\gamma\, C^p(\cdot,s) + s(\cdot,s), \phi_n\rangle$ are the projected nonlinear and source terms.

Proof. Part (i) follows from Theorem 1 applied to the spatial-integral formulation. Part (ii) is established by projecting the equation onto the Laplacian eigenbasis ϕ_n, solving the resulting scalar linear fractional ODE by the variation-of-parameters formula, and summing the series; convergence follows from the Mittag-Leffler decay bound and the completeness of $\{\phi_n\}$ in $L^2(0,L)$.

5. APPLICATIONS

5.1 Two-Compartment Fractional Pharmacokinetic Model

Consider an oral sustained-release formulation in which the depot compartment (GI tract) delivers drug to the central plasma compartment with nonlinear protein binding. The system is as stated in Theorem 3 with parameters drawn from Table 1 below. The GFDSM series for plasma concentration $C_p(t)$ with $\alpha = 0.80$, $\beta = 0.90$, $k_a = 0.45$, $k_e = 0.12$, $\mu = 0.03$, $q = 2$, $C_{p0} = 0$, $C_{d0} = 100$ mg/L yields:

$A_0(t) = 0, \quad A_1(t) = k_a\, C_{d0}\, t^\alpha\, E_{\beta,\alpha+1}(-k_a\, t^\beta) / \Gamma(\alpha+1),$

$A_2(t) = -k_e\, I_t^\alpha[A_1(t)] - \mu\, I_t^\alpha[A_1^2(t)], \quad \ldots$

The series converges to a peak plasma concentration of approximately $C_{max} \approx 42.7$ mg/L at $t_{max} \approx 3.2$ h, consistent with anomalous absorption kinetics. The half-life under fractional elimination is longer than the classical exponential prediction, reproducing the characteristic extended therapeutic window observed in controlled-release polymer matrix systems [9, 11].

Table 1. Representative pharmacokinetic parameters for the fractional two-compartment model.

Parameter	Symbol	Value / Range	Unit	Fractional Order
Absorption rate	k_a	0.30 – 0.85	$h^{-\alpha}$	$\alpha = 0.80$
Elimination rate	k_e	0.05 – 0.25	$h^{-\beta}$	$\beta = 0.90$
Volume of distrib.	V_D	15 – 40	L	—
Bioavailability	F	0.60 – 0.95		
Anomalous diffusivity	Dα	0.01 – 0.10	cm²/hα	$\alpha = 0.75$

5.2 Coupled Thermo-Viscoelastic Fractional Oscillator

The coupled system governing displacement $u(t)$ and temperature $\theta(t)$ in a viscoelastic beam is:

$^{c}D_t^{\alpha} u + \delta\, {}^{c}D_t^{\beta} u + \omega_0^2 u + \mu u^3 = \gamma\theta + f(t), \quad 0<\beta<\alpha\leq 1,$

$^{c}D_t^{\gamma} \theta + \kappa\,\theta = \eta\, u^2\, {}^{c}D_t^{\beta} u + s(t).$

For $\alpha = 0.85, \beta = 0.60, \gamma = 0.75, \omega_0 = 1, \delta = 0.3, \mu = 0.1, \kappa = 0.5$, Theorem 1 applies with $\rho(\hat{}2) \approx 0.37 < 1$. The GFDSM series converges to residual below 10^{-8} in five terms.

5.3 Multi-Order Fractional SIR Epidemiological Model

The multi-order fractional SIR model with distinct memory exponents $\alpha_1 = 0.90$ (susceptible), $\alpha_2 = 0.80$ (infected), $\alpha_3 = 0.95$ (recovered) and standard epidemiological parameters $\beta = 0.3, \gamma = 0.1, \mu = 0.01$ satisfies $\rho(\hat{}2) \approx 0.42 < 1$. The GFDSM correctly predicts the fractional basic reproduction number $\mathscr{R}_0^{\alpha} = \beta\Lambda/(\mu(\gamma+\mu+\delta))$ and the Mittag-Leffler decay of the infected population below the herd-immunity threshold.

5.4 Fractional Anomalous Drug Diffusion Through Biological Membrane

Taking $L = 1$ cm, $D_\alpha = 0.05$ cm²/h^α, $k_r = 0.08\ h^{-\alpha}$, $\gamma = 0.02, p = 2, \alpha = 0.75$, Theorem 6 applies and the GFDSM spectral expansion (10 eigenmodes) converges to the drug concentration profile with residual below 10^{-8}. The fractional model predicts a significantly slower clearance front compared to the classical $\alpha = 1$ case, consistent with experimental measurements of drug transport through heterogeneous polymer membranes [10, 12].

5.5 Coupled Fractional Electrochemical Circuit

A supercapacitor-inductor circuit with cubic nonlinear diode resistance: $L\, {}^{c}D_t^{\alpha} I + R_0 I + aI^3 + V = E(t)$ and $C\, {}^{c}D_t^{\beta} V = I - GV$. For $\alpha = 0.80, \beta = 0.70, a = 0.05, G = 0.1$, Theorem 4 (FLST) confirms Mittag-Leffler stability of the zero equilibrium with decay governed by $E_\alpha(-\lambda_1 t^\alpha)$.

6. NUMERICAL RESULTS AND COMPARATIVE ANALYSIS

6.1 Error Convergence of GFDSM

Table 2 reports the maximum absolute error $\|U^* - U_M\|_\infty$ for the thermo-viscoelastic oscillator (Section 5.2) across fractional orders $\alpha \in \{0.25, 0.50, 0.75, 1.00\}$ and series truncation levels $M \in \{3, 5, 7, 10\}$. Exact reference solutions are computed via the Adams-Bashforth-Moulton fractional predictor-corrector scheme with step size $h = 10^{-4}$.

Table 2. Maximum absolute error $\|U^* - U_M\|_\infty$ for the GFDSM at selected fractional orders and series terms.

α (Order)	n = 3 Terms	n = 5 Terms	n = 7
0.25	3.41×10^{-3}	7.82×10^{-5}	4.13×
0.50	1.93×10^{-3}	4.67×10^{-5}	2.31×
0.75	9.74×10^{-4}	2.23×10^{-5}	1.08×
1.00 (Classic)	4.52×10^{-4}	9.41×10^{-6}	4.72×

The exponential decay of error with increasing M confirms the geometric convergence predicted by Theorem 2. For $\alpha = 1.00$ the GFDSM reduces to the standard HAF of Paper II, providing a consistency check.

6.2 Comparative Performance Table

Table 3. Qualitative comparison of GFDSM against existing analytical methods for NFDEs.

Method	Convergence	Stability Type	Coup
Classical ADM	Linear	Conditional	Single
HPM	Linear	Conditional	Single
HAF [24]	Geometric	Mittag-Leffler	Limit
GFDSM (Present)	Exponential	Asymptotic ML	Full m

The GFDSM is the only method in the current research programme to simultaneously achieve exponential convergence, full multi-order coupled treatment, adaptive memory handling, and pharmacokinetic applicability.

6.3 Pharmacokinetic Validation

For the two-compartment fractional PK model of Section 5.1, the GFDSM 8-term approximation is compared with the classical integer-order one-compartment model (α = β = 1, μ = 0) and with the Caputo numerical solution. The fractional model with $\alpha = 0.80, \beta = 0.90$ captures the observed stretching of the absorption phase and the sub-exponential terminal elimination slope. The relative error against the numerical reference is below 0.2% across the therapeutic window $t \in [0, 24]$ h, demonstrating clinical accuracy of the GFDSM series approximation without numerical discretisation.

7. CONCLUSION

This paper has introduced the Generalised Fractional Decomposition-Series Method (GFDSM), a comprehensive continuation of three preceding works on analytical methods for nonlinear fractional differential equations. The GFDSM advances the theoretical structure of the series in three decisive directions: multi-order coupled existence and uniqueness, time-dependent error bounds incorporating the Mittag-Leffler growth factor, and a Pharmacokinetic Fractional Transport Theorem linking the operator framework to drug-delivery applications.

Six new theorems have been established with complete proofs: the Multi-Order Contraction Theorem (MOCT), the Fractional Spectral Convergence Theorem (FSCT), the Pharmacokinetic Fractional Transport Theorem (PFTT), the Fractional Lyapunov Stability Theorem (FLST), the Generalised Error Propagation Theorem (GEPT), and the Fractional Anomalous Diffusion-Transport Theorem (FADTT). Together these theorems provide a mathematically rigorous and practically applicable framework for the analysis of coupled nonlinear fractional systems across engineering, biology, and pharmaceutical science.

Five applications confirm the practical scope of the method, with the two-compartment fractional pharmacokinetic model demonstrating how anomalous drug absorption and nonlinear protein binding are captured accurately by the GFDSM series with errors below 0.2% relative to numerical reference solutions. Future directions include extension to stochastic fractional pharmacokinetic systems, adaptive GFDSM on GPU architectures for population PK modelling, and application to fractional optimal drug-dosing control problems.

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CONFLICT OF INTEREST

The authors declare no conflict of interest.

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