RESEARCH PAPER

Product and convolution theorems for fractional Hartley Transform

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ABSTRACT

This paper is concerned with the fractional Hartley transform of the product and convolution of two functions. Product and convolution theorems for the fractional Hartley transform are established, and corresponding results are derived. These theorems provide useful relationships between operations in the original domain and their representations in the transform domain. The obtained results extend the operational properties of fractional transforms and may have applications in signal processing and communication systems.

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1. INTRODUCTION

The Hartley transform is closely related to the Fourier transform and maps real-valued functions into real-valued functions. Introduced by R.V.L. Hartley as an alternative to the Fourier transform, it has the advantage of avoiding complex arithmetic while preserving many important analytical properties [5]. The Hartley transform is a real linear operator and possesses symmetric as well as self-inverse properties, which imply that the transform is unitary. Owing to these characteristics, the Hartley transform has found applications in signal processing, communication systems, and image analysis.

The concept of fractionalization of integral transforms has led to the development of the fractional Fourier transform, which generalizes the classical Fourier transform and provides intermediate representations between time and frequency domains. Numerous results and applications associated with the fractional Fourier transform have been established by several researchers. In particular, Almeida investigated product and convolution theorems for the fractional Fourier transform and demonstrated their importance in signal processing applications [1]. Zayed further developed convolution and product relations for the fractional Fourier transform [6], while multidimensional extensions of convolution and product theorems were studied in [3].

The fractional Hartley transform was introduced to extend the advantages of the Hartley transform into the fractional domain. Pei *et al.* proposed a continuous fractional Hartley transform and discussed its fundamental properties [4]. Later, Gaikwad introduced the fractional Hartley transform and its inverse and established several basic results associated with the transform [2]. However, operational properties such as convolution and product theorems for the fractional Hartley transform have received comparatively little attention.

Convolution and product operations are among the most fundamental and useful properties of integral transforms because of their significance in filtering, signal analysis, and system characterization. Motivated by analogous results available for the fractional Fourier transform [1, 6] and multidimensional fractional transforms [3], in this paper we derive the fractional Hartley transform of the usual product and convolution of two functions. The obtained results provide useful tools for simplifying computations in transformed domains and may contribute to further applications of the fractional Hartley transform.

2. THE FRACTIONAL FOURIER AND FRACTIONAL HARTLEY TRANSFORM

Definition 2.1 [5] If $h \in L^1(R) \cap C^1(R)$ *then the pair of Fourier transform and its inverse Fourier transform are given by*

$$F(v) = F[h](v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) e^{-jtv} dt$$

and

$$h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(v) e^{jtv} dv,$$

where $t,\ v \in \mathbb{R}$.

Definition 2.2 *[1] If* $h \in L^1(\mathbb{R}) \cap C^1(\mathbb{R})$, *and* $\alpha \in \mathbb{R}$; *with* α *is a constant then fractional Fourier transform are denoted by* $R^\alpha[h(t)](v)$ *or* $F_\alpha(v)$ *or* $g_F^\alpha(v)$ *and is defined by*

$$R^\alpha[h(t)](v) = F_\alpha(v) = g_F^\alpha(v) = \int_{-\infty}^{\infty} h(t) K_F^\alpha(t,\ v) dt$$

where $K_F^\alpha(t,\ v) = \sqrt{\frac{1 - j\cot\psi}{2\pi}}\ \exp\left\{ j\left[\frac{1}{2}(t^2 + v^2)\cot\psi - tv\csc\psi \right] \right\}$ *and*

$$h(t) = \int_{-\infty}^{\infty} \overline{K_F^\alpha(t,\ v)} F_\alpha(v) dv$$

where $\overline{K_F^\alpha(t,\ v)} = \sqrt{\frac{1 + j\cot\psi}{2\pi}}\ \exp\left\{ -j\left[\frac{1}{2}(t^2 + v^2)\cot\psi - tv\csc\psi \right] \right\}$.
and $\psi = \frac{\alpha\pi}{2}$ *if* $\psi \neq \pi m$, *for all* $m = 0, 1, 2, \ldots$

Definition 2.3 *[4], [5] If* $h \in L^1(R) \cap C^1(R)$, *then the pair of Hartley transform and its inverse Hartley transform are given by*

$$g_H(v) = g_H[h](v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) cas(tv) dt$$

and

$$h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_H(v) cas(tv) dv,$$

where $cas(t) = cos(t) + sin(t) = \sqrt{2}\ \sin(t + \frac{\pi}{4}) = \sqrt{2}\ \cos(t - \frac{\pi}{4})$.

Result 2.1 *[4] The relation between Hartly transform and Fourier transform is given by*

$$g_H(v) = \frac{1+i}{2} g_F(v) + \frac{1-i}{2} g_F(-v). \tag{2.1}$$

Result 2.2 *The relation between Fourier transform and Hartley transform is given by*

$$g_F(v) = \frac{1-i}{2} g_H(v) + \frac{1+i}{2} g_H(-v). \tag{2.2}$$

The proof is obvious by using the relation between Hartley and Fourier transform from above result.

Definition 2.4 *[4] If $h \in L^1(\mathbb{R}) \cap C^1(\mathbb{R}), t, v, \alpha \in \mathbb{R}^n$, where α is a constant, then the fractional Hartley transform of $h(t)$ is denoted by $g_H^\alpha(v)$ and defined as*

$$g_H^\alpha(v) = \int_{-\infty}^{\infty} K_H^\alpha(t, v)h(t)dt, \tag{2.3}$$

where

$$K_H^\alpha(t,\ v) = \sqrt{\frac{1 - i\cot\psi}{2\pi}}\ e^{i\frac{1}{2}(t^2+v^2)\cot\psi}[\cos(tv\csc\psi) + e^{i(\psi-\frac{\pi}{2})}\sin(tv\csc\psi)];$$

then $h(t)$ is given by

$$h(t) = \int_{-\infty}^{\infty} \overline{K_H^\alpha(t,\ v)}g_H^\alpha(v)dt,$$

where

$$\overline{K_H^\alpha(t,\ v)} = \sqrt{\frac{1 + i\cot\psi}{2\pi}}\ e^{-i\frac{t^2+v^2}{2}\cot\psi}[\cos(tv\csc\psi) + e^{-i(\psi-\frac{\pi}{2})}\sin(tv\csc\psi)]$$

and $\psi = \frac{\alpha\pi}{2}$ if $\psi \neq \pi m$, for all $m = 0, 1, 2, \ldots$

Note 2.1 *If $\psi = \frac{\pi}{2}$, then the extended transform defined in equation (2.3) reduces to Hartley transform.*

Result 2.3 *[4] The relation between kernel of fractional Hartley transform and kernel of fractional Fourier transform is written as*

$$K_H^\alpha(t, v) = \left[\frac{1 + e^{\frac{i\alpha\pi}{2}}}{2}\right]K_F^\alpha(t, v) + \left[\frac{1 - e^{\frac{i\alpha\pi}{2}}}{2}\right]K_F^\alpha(t, -v). \tag{2.4}$$

Result 2.4 *[2] If a kernel of one dimensional fractional Fourier transform is denoted by $K_F^\alpha(t, v)$ and a kernel of one dimensional fractional Hartley transform is denoted by $K_H^\alpha(t, v)$ then*

$$K_F^\alpha(t, v) = \left[\frac{1 + e^{\frac{-i\alpha\pi}{2}}}{2}\right]K_H^\alpha(t, v) + \left[\frac{1 - e^{\frac{-i\alpha\pi}{2}}}{2}\right]K_H^\alpha(t, -v)$$

Result 2.5 *[2] If $h \in L^1(\mathbb{R}) \cap C^1(\mathbb{R})$, the one dimensional fractional Fourier transform of $h(t)$ is denoted by $g_F^\alpha(v)$ and one dimensional fractional Hartley transform is denoted by $g_H^\alpha(v)$ then*

$$g_F^\alpha(v) = \left[\frac{1 + e^{\frac{-i\alpha\pi}{2}}}{2}\right]g_H^\alpha(v) + \left[\frac{1 - e^{\frac{-i\alpha\pi}{2}}}{2}\right]g_H^\alpha(-v).$$

Result 2.6 *[2] If $h \in L^1(\mathbb{R}) \cap C^1(\mathbb{R})$, the one dimensional fractional Fourier transform of $h(t)$ is denoted by $g_F^\alpha(v)$ and one dimensional fractional Hartley transform is denoted by $g_H^\alpha(v)$ then*

$$g_H^\alpha(v) = \left[\frac{1 + e^{\frac{i\alpha\pi}{2}}}{2}\right]g_F^\alpha(v) + \left[\frac{1 - e^{\frac{i\alpha\pi}{2}}}{2}\right]g_F^\alpha(-v).$$

Theorem 2.1 ***[1]*** **The transform of a Product**

Let us consider two functions $x, y \in \mathrm{L}^1(\mathbb{R}) \cap \mathcal{W}$ and let $z(t) = x(t)y(t)$;
where $\mathcal{W}$ is wiener algebra, involving the set of Fourier transforms of functions in $\mathrm{L}^1(\mathbb{R})$ and in the set of tempered distributions. The function z is in $\mathrm{L}^1(\mathbb{R})$ and thus its fractional Fourier transform Z_α is obtained by the definition of fractional Fourier transform. Let us compute $Z_\alpha(u)$ as follows:

$$Z_\alpha(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} X_\alpha(u - v\sin\alpha)Y(v)e^{-j\left(\frac{v^2}{2}\right)\sin\alpha\cos\alpha + juv\cos\alpha} dv.$$

Definition 2.5 *[5]* **Convolution** *If* $h, g \in \mathrm{L}^1(\mathrm{R}) \cap \mathrm{C}^1(\mathrm{R})$*, then their convolution f is denoted by h * g and defined as*

$$f(y) = [h * g](y) = \int_{-\infty}^{\infty} h(t)g(y - t)dt.$$

Definition 2.6 *[6] For any function $f(u)$, let us define the function, $\tilde{f}(u)$ by $\tilde{f}(u) = f(u)e^{ia(\psi)u^2}$, where $a(\psi) = \frac{cot\psi}{2}$, $c(\psi) = \sqrt{1 - i\cot\psi}$ and $\psi = \frac{\alpha\pi}{2}$.*
We define the convolution operation $\star$ for any two functions f and g by,

$$h(u) = (f \star g)(u) = \frac{c(\psi)}{\sqrt{2\pi}} e^{-ia(\psi)u^2} (\tilde{f} * \tilde{g})(u).$$

Theorem 2.2 ***[6]*** **Convolution Theorem for the fractional Fourier transform**

Let $h(u) = (f \star g)(u)$ and $g^\alpha_{F_1}(w), g^\alpha_{F_2}(w), g^\alpha_{F_3}(w)$ denotes the fractional Fourier transform of f, g, h respectively. Then

$$g^\alpha_{F_3}(w) = g^\alpha_{F_1}(w) g^\alpha_{F_2}(w) e^{-ia(\psi)w^2}. \tag{2.5}$$

3. FRACTIONAL HARTLEY TRANSFORM OF USUAL PRODUCT AND CONVOLU-TION

3.1 The transform of a product

If two functions x, y are in $L^1(\mathbb{R}) \cap C^1(\mathbb{R})$ and let $z(t) = x(t)y(t)$,
then function z is in $L^1(\mathbb{R}) \cap C^1(\mathbb{R})$. Also let Z_F^α and Z_H^α are denoted by fractional Fourier transform of z and fractional Hartley transform of z respectively. The function z is in $L^1(\mathbb{R}) \cap C^1(\mathbb{R})$ and thus its fractional Hartley transform Z_H^α is obtained by the definition. Let us compute $Z_H^\alpha(u)$, as follows [1]:

$$
\begin{aligned}
&Z_H^\alpha(u) \\
&= \left[\frac{1+e^{i\psi}}{2}\right] Z_F^\alpha(u) + \left[\frac{1-e^{i\psi}}{2}\right] Z_F^\alpha(-u) \\
&= \left[\frac{1+e^{i\psi}}{2}\right]\left(\frac{|\csc\psi|}{\sqrt{2\pi}} e^{j\left(\frac{u^2}{2}\right)\cot\psi}\right)\int_{-\infty}^{\infty} \mathrm{X}_{\mathrm{F}}^\alpha(v) e^{-j\left(\frac{v^2}{2}\right)\cot\psi} Y_F[y][(u-v)\csc\psi]dv + \\
&\quad \left[\frac{1-e^{i\psi}}{2}\right]\left(\frac{|\csc\psi|}{\sqrt{2\pi}} e^{j\left(\frac{u^2}{2}\right)\cot\psi}\right)\int_{-\infty}^{\infty} \mathrm{X}_{\mathrm{F}}^\alpha(-v) e^{-j\left(\frac{v^2}{2}\right)\cot\psi} Y_F[y][-(u-v)\csc\psi]dv
\end{aligned}
$$

where $Y_F[y]$ is the Fourier transform of y.

Let us denote $A = \left(\frac{|\csc\psi|}{\sqrt{2\pi}} e^{j\left(\frac{u^2}{2}\right)\cot\psi}\right)$ and $B = e^{-j\left(\frac{v^2}{2}\right)\cot\psi}$ and use Fourier transform and fractional Fourier transform in terms of Hartley transform and fractional Hartley transform respectively then above equation becomes

$$
\begin{aligned}
&Z_H^\alpha(u) \\
&= \left(\frac{1+e^{i\psi}}{2}\right) A \int_{-\infty}^{\infty} \left\{\left(\frac{1+e^{-i\psi}}{2}\right)\mathrm{X}_{\mathrm{H}}^\alpha(v) + \left(\frac{1-e^{-i\psi}}{2}\right)\mathrm{X}_{\mathrm{H}}^\alpha(-v)\right\} B \left\{\left(\frac{1-i}{2}\right) Y_H[y][(u-v)\csc\psi] + \right. \\
&\quad \left.\left(\frac{1+i}{2}\right) Y_H[y][-(u-v)\csc\psi]\right\} dv + \left(\frac{1-e^{i\psi}}{2}\right) A \int_{-\infty}^{\infty} \left\{\left(\frac{1+e^{-i\psi}}{2}\right)\mathrm{X}_{\mathrm{H}}^\alpha(-v) + \right. \\
&\quad \left.\left(\frac{1-e^{-i\psi}}{2}\right)\mathrm{X}_{\mathrm{H}}^\alpha(v)\right\} B \left\{\left(\frac{1-i}{2}\right) Y_H[y][-(u-v)\csc\psi] + \left(\frac{1+i}{2}\right) Y_H[y][(u-v)\csc\psi]\right\} dv
\end{aligned}
$$

$$
\begin{aligned}
&Z_H^\alpha(u)\\
&= \left(\frac{1+e^{i\psi}}{2}\right)\left(\frac{1+e^{-i\psi}}{2}\right)\left(\frac{1-i}{2}\right)A\int_{-\infty}^{\infty} \mathrm{X}_{\mathrm{H}}^\alpha(v)BY_H[y][(u-v)\csc\psi]dv\\
&\quad+\left(\frac{1+e^{i\psi}}{2}\right)\left(\frac{1+e^{-i\psi}}{2}\right)\left(\frac{1+i}{2}\right)A\int_{-\infty}^{\infty} \mathrm{X}_{\mathrm{H}}^\alpha(v)BY_H[y][-(u-v)\csc\psi]dv\\
&\quad+\left(\frac{1+e^{i\psi}}{2}\right)\left(\frac{1-e^{-i\psi}}{2}\right)\left(\frac{1-i}{2}\right)A\int_{-\infty}^{\infty} \mathrm{X}_{\mathrm{H}}^\alpha(-v)BY_H[y][(u-v)\csc\psi]dv\\
&\quad+\left(\frac{1+e^{i\psi}}{2}\right)\left(\frac{1-e^{-i\psi}}{2}\right)\left(\frac{1+i}{2}\right)A\int_{-\infty}^{\infty} \mathrm{X}_{\mathrm{H}}^\alpha(-v)BY_H[y][-(u-v)\csc\psi]dv\\
&\quad+\left(\frac{1-e^{i\psi}}{2}\right)\left(\frac{1+e^{-i\psi}}{2}\right)\left(\frac{1-i}{2}\right)A\int_{-\infty}^{\infty} \mathrm{X}_{\mathrm{H}}^\alpha(-v)BY_H[y][-(u-v)\csc\psi]dv\\
&\quad+\left(\frac{1-e^{i\psi}}{2}\right)\left(\frac{1+e^{-i\psi}}{2}\right)\left(\frac{1+i}{2}\right)A\int_{-\infty}^{\infty} \mathrm{X}_{\mathrm{H}}^\alpha(-v)BY_H[y][(u-v)\csc\psi]dv\\
&\quad+\left(\frac{1-e^{i\psi}}{2}\right)\left(\frac{1-e^{-i\psi}}{2}\right)\left(\frac{1-i}{2}\right)A\int_{-\infty}^{\infty} \mathrm{X}_{\mathrm{H}}^\alpha(v)BY_H[y][-(u-v)\csc\psi]dv\\
&\quad+\left(\frac{1-e^{i\psi}}{2}\right)\left(\frac{1-e^{-i\psi}}{2}\right)\left(\frac{1+i}{2}\right)A\int_{-\infty}^{\infty} \mathrm{X}_{\mathrm{H}}^\alpha(v)BY_H[y][(u-v)\csc\psi]dv
\end{aligned}
$$

$$
\begin{aligned}
&Z_H^\alpha(u)\\
&= \frac{1}{2}(1-i\cos\psi)A\int_{-\infty}^{\infty} \mathrm{X}_{\mathrm{H}}^\alpha(v)BY_H[y][(u-v)\csc\psi]dv\\
&\quad+\frac{1}{2}(1+i\cos\psi)A\int_{-\infty}^{\infty} \mathrm{X}_{\mathrm{H}}^\alpha(v)BY_H[y][-(u-v)\csc\psi]dv\\
&\quad+\frac{1}{2}(\sin\psi)A\int_{-\infty}^{\infty} \mathrm{X}_{\mathrm{H}}^\alpha(-v)BY_H[y][(u-v)\csc\psi]dv\\
&\quad+\frac{-1}{2}(\sin\psi)A\int_{-\infty}^{\infty} \mathrm{X}_{\mathrm{H}}^\alpha(-v)BY_H[y][-(u-v)\csc\psi]dv
\end{aligned}
$$

Theorem 3.1 Convolution theorem for the fractional Hartley transform

Let $h(u) = (f \star g)(u)$, where $\star$ is defined as in definition (2.6) and $g_{F_1}^\alpha(w)$, $g_{F_2}^\alpha(w)$, $g_{F_3}^\alpha(w)$ denotes the fractional Fourier transform of f, g, h respectively and $g_{H_1}^\alpha(w), g_{H_2}^\alpha(w), g_{H_3}^\alpha(w)$ denotes the fractional Hartley transform of f, g, h respectively. Then

$$
\begin{aligned}
g_{H_3}^\alpha(w) = \frac{e^{-ia(\psi)w^2}}{4}\Big\{&(3+e^{-2i\psi})g_{H_1}^\alpha(w)g_{H_2}^\alpha(w) + (1-e^{-2i\psi})g_{H_1}^\alpha(w)g_{H_2}^\alpha(-w)\\
&+(1-e^{-2i\psi})g_{H_1}^\alpha(-w)g_{H_2}^\alpha(w) - (1-e^{-2i\psi})g_{H_1}^\alpha(-w)g_{H_2}^\alpha(-w)\Big\}
\end{aligned}
$$

Proof: By utilizing the definition of the fractional Fourier transform and the established relationship between the fractional Hartley transform and the fractional Fourier transform, along with their corresponding inverse operations, we can elucidate a significant result. Furthermore, the application of the convolution theorem for the fractional Fourier transform, as elucidated by Zayed [6], consequently, we derive

$$g_{H_3}^{\alpha}(w) = \int_{-\infty}^{\infty} K_H^{\alpha}(u,w)h(u)du$$

$$= \int_{-\infty}^{\infty} \left\{ \left(\frac{1+e^{i\psi}}{2}\right) K_F^{\alpha}(u,w) + \left(\frac{1-e^{i\psi}}{2}\right) K_F^{\alpha}(u,-w) \right\} h(u)dt$$

Also

$$g_{H_3}^{\alpha}(w) = \left(\frac{1+e^{i\psi}}{2}\right) g_{F_3}^{\alpha}(w) + \left(\frac{1-e^{i\psi}}{2}\right) g_{F_3}^{\alpha}(-w)$$

$$= \left(\frac{1+e^{i\psi}}{2}\right) e^{-ia(\psi)w^2} g_{F_1}^{\alpha}(w) g_{F_2}^{\alpha}(w) + \left(\frac{1-e^{i\psi}}{2}\right) e^{-ia(\psi)w^2} g_{F_1}^{\alpha}(-w) g_{F_2}^{\alpha}(-w)$$

$$= \left(\frac{1+e^{i\psi}}{2}\right) e^{-ia(\psi)w^2} \left\{ \left[\left(\frac{1+e^{-i\psi}}{2}\right) g_{H_1}^{\alpha}(w) + \left(\frac{1-e^{-i\psi}}{2}\right) g_{H_1}^{\alpha}(-w) \right] \times \right.$$
$$\left. \left[\left(\frac{1+e^{-i\psi}}{2}\right) g_{H_2}^{\alpha}(w) + \left(\frac{1-e^{-i\psi}}{2}\right) g_{H_2}^{\alpha}(-w) \right] \right\}$$
$$+ \left(\frac{1-e^{i\psi}}{2}\right) e^{-ia(\psi)w^2} \left\{ \left[\left(\frac{1+e^{-i\psi}}{2}\right) g_{H_1}^{\alpha}(-w) + \left(\frac{1-e^{-i\psi}}{2}\right) g_{H_1}^{\alpha}(w) \right] \times \right.$$
$$\left. \left[\left(\frac{1+e^{-i\psi}}{2}\right) g_{H_2}^{\alpha}(-w) + \left(\frac{1-e^{-i\psi}}{2}\right) g_{H_2}^{\alpha}(w) \right] \right\}$$

$$= \frac{e^{-ia(\psi)w^2}}{4} \left[\left(1 + e^{i\psi} + e^{-i\psi} + 1\right) g_{H_1}^{\alpha}(w) + (1 + e^{i\psi} - e^{-i\psi} - 1) g_{H_1}^{\alpha}(-w) \right] \times$$
$$\left[\left(\frac{1+e^{-i\psi}}{2}\right) g_{H_2}^{\alpha}(w) + \left(\frac{1-e^{-i\psi}}{2}\right) g_{H_2}^{\alpha}(-w) \right] + \frac{e^{-ia(\psi)w^2}}{4} \times$$
$$\left[(1 - e^{i\psi} + e^{-i\psi} - 1) g_{H_1}^{\alpha}(-w) + (1 + e^{i\psi} - e^{-i\psi} + 1) g_{H_1}^{\alpha}(w) \right] \times$$
$$\left[\left(\frac{1+e^{-i\psi}}{2}\right) g_{H_2}^{\alpha}(-w) + \left(\frac{1-e^{-i\psi}}{2}\right) g_{H_2}^{\alpha}(w) \right]$$

$$= \frac{e^{-ia(\psi)w^2}}{8} \left\{ (2 + e^{i\psi} + e^{-i\psi})(1 + e^{-i\psi}) g_{H_1}^{\alpha}(w) g_{H_2}^{\alpha}(w) + (2 + e^{i\psi} + e^{-i\psi}) \times \right.$$
$$(1 - e^{-i\psi}) g_{H_1}^{\alpha}(w) g_{H_2}^{\alpha}(-w) + (e^{i\psi} - e^{-i\psi})(1 + e^{-i\psi}) g_{H_1}^{\alpha}(-w) g_{H_2}^{\alpha}(w)$$
$$\left. + (e^{i\psi} - e^{-i\psi})(1 - e^{-i\psi}) g_{H_1}^{\alpha}(-w) g_{H_2}^{\alpha}(-w) \right\} + \frac{e^{-ia(\psi)w^2}}{8} \times$$
$$\left\{ (2 - e^{i\psi} + e^{-i\psi})(1 + e^{-i\psi}) g_{H_1}^{\alpha}(w) g_{H_2}^{\alpha}(-w) + (2 - e^{i\psi} - e^{-i\psi}) \times \right.$$
$$(1 - e^{-i\psi}) g_{H_1}^{\alpha}(w) g_{H_2}^{\alpha}(w) + (-e^{i\psi} + e^{-i\psi})(1 + e^{-i\psi}) g_{H_1}^{\alpha}(-w) g_{H_2}^{\alpha}(-w) +$$
$$\left. (-e^{i\psi} + e^{-i\psi})(1 - e^{-i\psi}) g_{H_1}^{\alpha}(-w) g_{H_2}^{\alpha}(w) \right\}$$

$$g^{\alpha}_{H_3}(w) = \frac{e^{-ia(\psi)w^2}}{8}\Big\{(2 + e^{i\psi} + e^{-i\psi} + 2e^{-i\psi} + 1 + e^{-2i\psi})g^{\alpha}_{H_1}(w)g^{\alpha}_{H_2}(w) + (2 + e^{i\psi} + e^{-i\psi} - 2e^{-i\psi} - 1 - e^{-2i\psi})g^{\alpha}_{H_1}(w)g^{\alpha}_{H_2}(-w) + (e^{i\psi} - e^{-i\psi} + 1 - e^{-2i\psi}) g^{\alpha}_{H_1}(-w)g^{\alpha}_{H_2}(w) + (e^{i\psi} - e^{-i\psi} - 1 + e^{-2i\psi})g^{\alpha}_{H_1}(-w)g^{\alpha}_{H_2}(-w)\Big\} + \frac{e^{-ia(\psi)w^2}}{8}\Big\{(2 - e^{i\psi} - e^{-i\psi} + 2e^{-i\psi} - 1 - e^{-2i\psi})g^{\alpha}_{H_1}(w)g^{\alpha}_{H_2}(-w) + (2 - e^{i\psi} - e^{-i\psi} - 2e^{-i\psi} + 1 + e^{-2i\psi})g^{\alpha}_{H_1}(w)g^{\alpha}_{H_2}(w) + (-e^{i\psi} + e^{-i\psi} - 1 - e^{-2i\psi})g^{\alpha}_{H_1}(-w)g^{\alpha}_{H_2}(-w) + (-e^{i\psi} + e^{-i\psi} + 1 - e^{-2i\psi})g^{\alpha}_{H_1}(-w)g^{\alpha}_{H_2}(w)\Big\}$$

$$g^{\alpha}_{H_3}(w) = \frac{e^{-ia(\psi)w^2}}{4}\Big\{(3 + e^{-2i\psi})g^{\alpha}_{H_1}(w)g^{\alpha}_{H_2}(w) + (1 - e^{-2i\psi})g^{\alpha}_{H_1}(w)g^{\alpha}_{H_2}(-w) + (1 - e^{-2i\psi})g^{\alpha}_{H_1}(-w)g^{\alpha}_{H_2}(w) - (1 - e^{-2i\psi})g^{\alpha}_{H_1}(-w)g^{\alpha}_{H_2}(-w)\Big\}$$

Remark 3.1 *In this paper we have defined the kernel and integral transform for those* $\alpha \in R$ *such that* α *is not multiple of* π. *We could not consider the case where* α *is multiple of* π.

4. CONCLUSION

In this paper, product and convolution theorems associated with the fractional Hartley trans-form have been established. The fractional Hartley transform of the usual product and convolution of two functions has been derived, and corresponding relationships in the transform domain have been obtained. These results extend important operational properties of integral transforms to the fractional Hartley transform and provide useful tools for simplifying computations involving transformed functions. The obtained theorems contribute to the theoretical development of the fractional Hartley transform and may have potential applications in signal processing, communication systems, filtering, and related areas. Further investigations may focus on multidimensional extensions and applications of these results in engineering and applied sciences.

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