RESEARCH PAPER

Analytical determination of the shadow of the schwarzschild black hole

Fulgencio Villegas Silva[1], Miguel Castillo Corzo[2], and Marco A. Merma Jara[3*]

[1,2,3]Facultad de Ciencias Físicas, Universidad Nacional Mayor de San Marcos, Lima, Perú
*[*]**Corresponding author:** Marco Merma; Facultad de Ciencias Físicas, Universidad Nacional Mayor de San Marcos, Lima, Perú*
aulaslivres@gmail.com

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ABSTRACT

Objective: This paper undertakes an analytical study of the shadow of the Schwarzschild black hole.
Method: In order to accomplish this objective, it is necessary to emphasize the definition of a black hole, its primary characteristics, and its significance in the field of cosmology. In this case, the shadow of an object is defined as the shape that the object possesses in space, as perceived by an observer when considering solely the properties of spacetime.
Results: The analytical solution is derived using the Hamilton-Jacobi equation, the method of separation of variables, and the introduction of a Carter-type conserved quantity.
Conclusions: The determination of constant-radius orbits is achieved through the utilization of impact parameters and celestial coordinates.

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1 INTRODUCTION

A black hole is a finite region of space-time connected to an enormous density of matter, which generates a very intense gravitational field; beyond a certain limit, there is no possibility for any type of matter to escape its pull. Its existence as such was described by Karl Schwarzschild's solution to Einstein's equations in his theory of general relativity.

The analysis of the shadow of black holes is of great importance in the cosmological field; its observation allows to determine the mass and spin of a black hole [1,2]. In 1993, Cunningham and Bardeen made use, for the first time, of the visual analysis of black holes to study the movement of a star around a black hole [3]. However, the first work of visual analysis of a black hole made by computer was carried out by Luminet in 1979 [4], as shown in Figure 1.

In 2019, using the Event Horizon Telescope, the first image of the supermassive black hole at the center of the galaxy Messier 87 (M87) located in the constellation of Virgo was obtained [5,6,7,8,9,10], as shown in Figure 2. This black hole is approximately 6.5 billion times more massive than our Sun and is located at a distance of 55 million light-years from Earth. This first image constitutes an important milestone in cosmology, on the one hand, it shows us evidence of the existence of black holes, and on the other hand, it allows us to understand the shadow of a black hole. These results reveal a fine structure near the black hole horizon.

The shadow of a black hole is a two-dimensional dark region relative to an observer's sky and is caused by light rays falling on the event horizon due to the black hole's intense gravity. The shadow of a black hole was first studied in 1966 by Synge [11]. In general, the shadow cast by a non-rotating black hole has the standard circular shape, while a rotating black hole has a deformed circular shadow, elongated in the direction of the spin axis [12,13].

A black hole is described by its mass, angular momentum, and electric charge. In the case of supermassive black holes, their gravitational field is basically described by their mass and angular momentum since they are usually electrically neutral. The shadow of a black hole appears as a dark region surrounded by a sphere of photons [15] that is concentric with the gravitational object. The intensity of the black hole's gravitational field deflects light rays. If we project a ray of light towards a black hole, it bends, leaning towards the black hole and forming part of the spherical surface above the event horizon.

**Author for Correspondence:* aulaslivres@gmail.com

2 EINSTEIN'S FIELD EQUATIONS

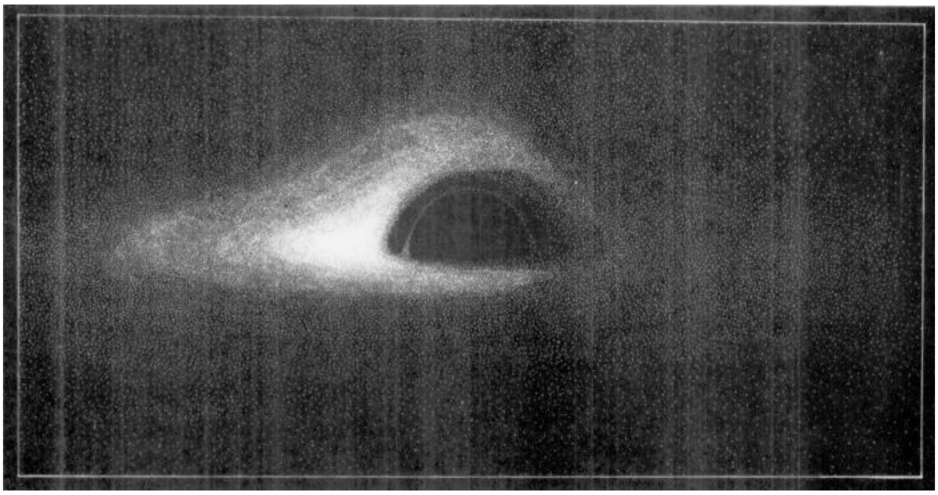

Figure 1: Simulated image of a black hole surrounded by a thin accretion disk [14].

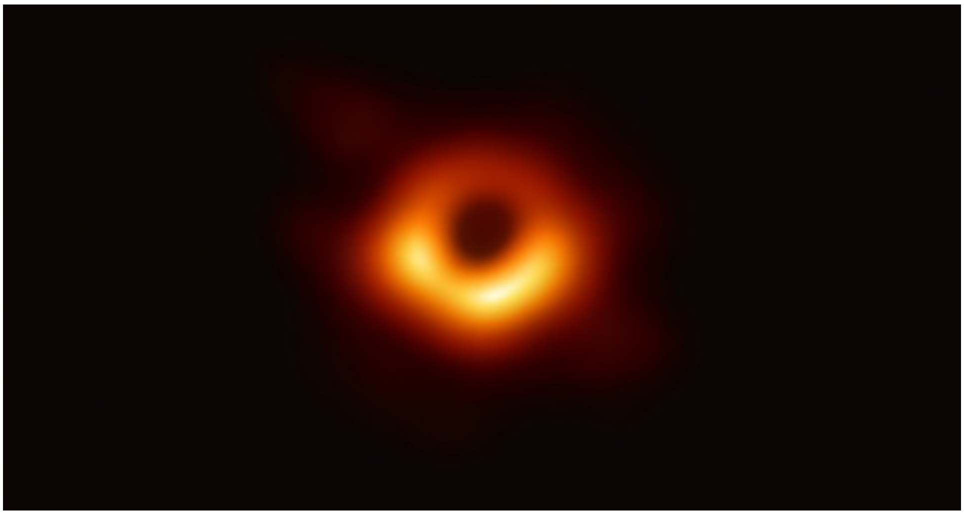

Figure 2: First image of a black hole located at the center of the galaxy M87. The accretion disk can be seen as a luminous region surrounding the black hole [14].

Einstein's field equations provide a mathematical description of the theory of general relativity; that is, they relate the presence of matter to the curvature of space-time.

Einstein's field equations can be derived from the Lagrangian formalism, which is based on the principle of least action. Using variational calculus, the action is related to the gravitational field; it is also known that for the minimum, the action must be stationary. This proposition is represented by the equation (eq. 1):

$$\delta S = 0. \qquad (1)$$

The action for general relativity was formulated by Hilbert in 1915 and is called the Einstein-Hilbert action, which is given by equation (2) [16]:

$$S_G = \frac{1}{2k}\int_M L\left(g_{\mu\nu}\right)\sqrt{-g}\, d^4dx, \qquad (2)$$

where M is the spacetime completely covered by the integral; $g_{\mu\nu}$ is the metric tensor with determinant g; and k is a constant chosen so that the weak field limit reproduces Newtonian gravity.

Proposing a Lagrangian density defined as:

$$L\left(g_{\mu\nu}\right) = R, \qquad (3)$$

where R is the Ricci scalar. Therefore, the action is given by:

$$S_G = \frac{1}{2k}\int_M \left(R_{\mu\nu} g^{\mu\nu}\sqrt{-g}\right) d^4x. \qquad (4)$$

Considering the variation of the action with respect to the metric, equation (4) results in:

$$\delta S_G = \frac{1}{2k}\int_M \left[\delta\left(R_{\mu\nu}\right) g^{\mu\nu}\sqrt{-g} + R_{\mu\nu}\delta\left(g^{\mu\nu}\sqrt{-g}\right)\right] d^4x. \qquad (5)$$

Taking into account the following relationships:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu} \qquad (6)$$

$$\int_M \delta\left(R_{\mu\nu}\right) g^{\mu\nu}\sqrt{-g}\, d^4x = 0. \qquad (7)$$

Replacing equations (6) and (7) in (5), we obtain:

$$\delta S_G = \frac{1}{2k}\int \left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\sqrt{-g}\delta g^{\mu\nu}\, d^4x. \qquad (8)$$

This integral is zero for any variation $\delta g^{\mu\nu}$. This indicates that the integrand is zero, therefore it follows that [17]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \qquad (9)$$

These are Einstein's equations in a vacuum. For a space-time with the presence of matter or energy, Einstein's equations take the following form [18]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}, \qquad (10)$$

where $T_{\mu\nu}$ is the energy-momentum tensor.

3 SCHWARZSCHILD METRIC

The first exact solution to Einstein's field equations was obtained by Karl [19]. Schwarzschild shortly after Einstein published his General Theory of Relativity (GTR) [20]. This solution is for a static system, endowed with spherical symmetry and in vacuum, and presents an irremovable singularity known as the Schwarzschild black hole.

The Schwarzschild metric is obtained by solving the Einstein field equations, in coordinates(r, θ, ϕ, t), Its most common expression is:

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \frac{dr^2}{1-\frac{2m}{r}} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \qquad (11)$$

where m is a constant that is related to the mass of the body that generates the gravitational field.

The Schwarzschild metric is the simplest black hole solution to Einstein's equation. Equation (11) exhibits singularities at $r = 0$ and $r = 2m$; However, if the components of the Riemann tensor are calculated, it is found that for $r = 0$ they are not well defined, while for $r = 2m$ they are.

This indicates that the singularity associated with the value $r = 2m$ is fictitious, called a removable singularity, while for the value $r = 0$ it is an essential singularity. Choosing appropriate coordinates can make the singularity disappear at $r = 2m$. However, in the definition of the coordinates we chose, the singularity corresponds to the event horizon. That is:

$$r_{\text{horizon}} = 2m.$$

It is also observed that at the limit $r \to \infty$, Equation (11) reduces to the Minkowski metric in spherical coordinates, so the Schwarzschild metric is an asymptotically flat solution.

4 CELESTIAL COORDINATES

Light rays are bent by the black hole's gravitational field. The path of interest is the one that allows the light to escape and reach Earth. The most appropriate way to represent a distant observer is through celestial coordinates. The shadow of the black hole is described by the graph relating these coordinates.

The use of celestial coordinates requires that the metric be asymptotically flat, so that a very distant observer can consider a Euclidean reference system with coordinates (x, y, z), where the axis of rotation is along the z-axis, and the observer is located at the point $(r_0, \theta_0, 0)$ with $r_0 \to \infty$. The trajectory of a light ray, relative to the observer, can be described by the parametric curve $x(r), y(r), z(r)$ where $r^2 = x^2 + y^2 + z^2$. With respect to the observer, the tangent vector $\vec{T}$ to the parametric curve is given by [21]

$$\vec{T} = \left(\frac{dx}{dr}\right)\bigg|_{r_o}\hat{x} + \left(\frac{dy}{dr}\right)\bigg|_{r_o}\hat{y} + \left(\frac{dz}{dr}\right)\bigg|_{r_o}\hat{z} \qquad (12)$$

The line containing the tangent vector cuts the plane at the point(α_i, β_i) as shown in Figure 3.

The point (α_i, β_i) on the plane it is $(-\beta_i\, cos\, \theta_o, \alpha_i, \beta_i\, sin\, \theta_o)$, making the transformation to spherical coordinates (r, θ, ϕ), It is shown in Figure 3 [22]

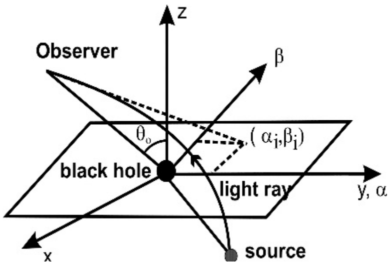

Figure 3: The angles α and β and the celestial coordinates describe the shadow of the black hole [21].

$$\alpha_i = -r_o^2 sen\theta_o \left.\frac{d\phi}{dr}\right|_{r_o} \qquad (13)$$

$$\beta_i = r_o^2 \left.\frac{d\theta}{dr}\right|_{r_o} \qquad (14)$$

5 SHADOW OF THE SCHWARZSCHILD BLACK HOLE

The shadow of the Schwarzschild black hole is determined by the Hamilton-Jacobi formalism.

The Lagrangian is given by [23]

$$L(x^\mu, \dot{x}^\mu) = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \quad \mu,\nu \in \{0,1,2,3\} = \{t, r, \theta, \phi\}, \qquad (15)$$

where $g_{\mu\nu}$ is the metric tensor and $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$, where λ is the affine parameter [24]

The Hamiltonian$H(x^\mu, p_\mu)$ is defined by [25]

$$H(x^\mu, p_\mu) = \dot{x}^\mu p_\mu - L(x^\mu, \dot{x}^\mu), \qquad (16)$$

where p_μ is given by:

$$p_\alpha = \frac{\partial L}{\partial \dot{x}^\alpha} = g_{\alpha\mu}\dot{x}^\mu. \qquad (17)$$

Replacing (15) in (16), we obtain:

$$H\left(x^{\mu}, p_{\mu}\right) = \frac{1}{2} g^{\mu\nu} p_{\mu} p_{\nu}. \qquad (18)$$

Because $H\left(x^{\mu}, p_{\mu}\right)$ is a constant of movement, it is possible to do the normalization:

$H\left(x^{\mu}, p_{\mu}\right) = -m^2$, where m is the mass of the particle at rest.

The Hamilton-Jacobi equation has the form:

$$H\left(x^{\mu}, p_{\mu}\right) = \frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^{\mu}} \frac{\partial S}{\partial x^{\nu}} = -m^2, \quad p_{\mu} = \frac{\partial S}{\partial x^{\mu}}, \qquad (19)$$

where S is the principal Hamiltonian function, also called the action.

Using equation (19), we obtain:

$$\frac{1}{2}\left[g^{tt}\left(\frac{\partial S}{\partial t}\right)^2 + g^{rr}\left(\frac{\partial S}{\partial r}\right)^2 + \mathrm{g}^{\theta\theta}\left(\frac{\partial S}{\partial \theta}\right)^2 + \mathrm{g}^{\phi\phi}\left(\frac{\partial S}{\partial \phi}\right)^2\right] = -m^2. \qquad (20)$$

Since this is the Schwarzschild metric, being spherically symmetric and static, any particle within this space-time presents a conservation of its energy E and its angular momentum L. To solve the Hamilton-Jacobi equation given by equation (21), the method of separation of variables is used, for which a principal Hamiltonian function of the form is proposed

$$S(x^{\mu}) = -Et + L\phi + S_r(r) + S_{\theta}\theta(\theta\theta) + \frac{1}{2} m^2 \lambda. \qquad (21)$$

Since our interest is to analyze the shadow of the Schwarzschild black hole, that is, we are interested in the trajectory of the light particles or photons. Therefore, $m = 0$ is considered in the above equations. Replacing equation (22) in equation (21), we obtain

$$\left(-\frac{1}{1-\frac{2M}{r}}\right)E^2 + \left(1 - \frac{2M}{r}\right)\left(\frac{dS_r}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dS_{\theta}}{d\theta}\right)^2 + \frac{L^2}{r^2 \mathrm{sen}^2\theta} = 0. \qquad (22)$$

By separating the radial and angular variables, we deduce:

$$\left(-\frac{r^2}{1-\frac{2M}{r}}\right)E^2 + r^2\left(1 - \frac{2M}{r}\right)\left(\frac{dS_r}{dr}\right)^2 = C, \qquad (23)$$

$$\left(\frac{dS_{\theta}}{d\theta}\right)^2 + \frac{L^2}{\sin^2\theta} = C, \qquad (24)$$

where C is Carter's constant [26].

In order to reduce the number of variables determining the trajectory of photons in the vicinity of the Schwarzschild black hole, we define the following variables called impact variables:

$$\gamma = \frac{L}{E}, \quad \sigma = \frac{C}{E^2}, \quad \bar{\lambda} = E\lambda. \qquad (25)$$

On the other hand, the above equations can be written in terms of $\frac{dr}{d\lambda}$ and $\frac{d\theta}{d\lambda}$ and in terms of the impact parameters, obtaining:

$$\frac{dt}{d\lambda} = -\frac{\gamma}{1-\frac{2M}{r}}, \qquad (26)$$

$$r^2 \frac{dr}{d\lambda} = \sqrt{R}, \qquad (27)$$

$$r^2 \frac{d\theta}{d\lambda} = \sqrt{\Theta}, \qquad (28)$$

$$\frac{d\phi}{d\lambda} = \frac{\gamma}{r^2 \sin^2\theta}. \qquad (29)$$

As the orbits have constant radii, they lead to considering critical values in the impact parameters, which leads to considering $R = 0$ and $\frac{dR}{dr} = 0$, getting:

$$2r^3 - r\sigma + M\sigma = 0, \qquad (30)$$

$$r^3 - r\sigma + 2M\sigma = 0. \qquad (31)$$

Solving equations (33) and (34), we obtain:

$$r = 3M, \qquad (32)$$

Replacing equation (35) in equations (33) or (34), we obtain:

$$\sigma = 27M^2. \qquad (33)$$

Using celestial coordinates, which are defined by:

$$\alpha = -r_0^2 \sin\theta_0 \frac{d\phi}{dr}|_{r_0\to\infty}, \qquad (34)$$

$$\beta = r_0^2 \frac{d\theta}{dr}|_{r_0\to\infty}, \qquad (35)$$

Replacing in equations (37) and (38) the relations $\frac{d\theta}{d\lambda}$ and $\frac{d\phi}{d\lambda}$ corresponding to the trajectories of the photons or geodesics, we obtain:

$$\alpha = -\frac{\gamma}{\sin\theta}, \qquad (36)$$

$$\beta = \sqrt{\sigma - \frac{\gamma^2}{\sin^2\theta}}. \qquad (37)$$

Eliminating the γ term from equations (39) and (40), the following relationship is found:

$$\alpha^2 + \beta^2 = 27M^2. \qquad (38)$$

Equation (41) represents the expression for the Schwarzschild black hole shadow, which can be graphed for different values of M, as shown in Figure 4.

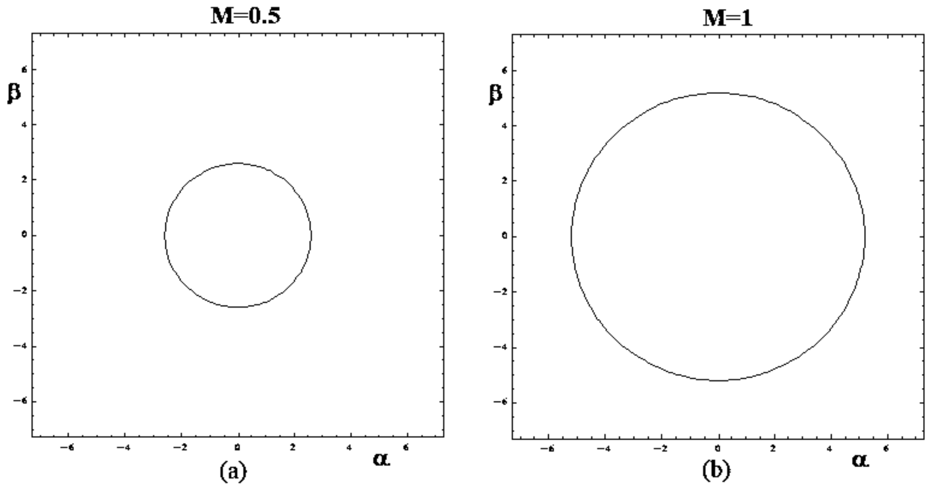

Figure 4: Shadow of the Schwarzschild black hole. M = 0.5 in figure (a) and M = 1 in figure (b).

From equation (41), we can notice that the shadow of the black hole is proportional to its mass and is represented, in space $\alpha - \beta$, by a circle of radius $\sqrt{27}M$.

6 CONCLUSIONS

The shadow of the Schwarzschild black hole has been studied by the motion of a test particle and the null geodesic equations have been derived by applying the Hamilton-Jacobi equation and the method of separation of variables.

The Hamilton-Jacobi equations have been shown to be separable, and it has therefore been possible to determine the silhouette of the Schwarzschild black hole's shadow.

Our result indicates that the shadow of the Schwarzschild black hole is circular, with a radius of $\sqrt{27}M$. Thus, the shadow's size increases with its mass.

Because the Schwarzschild metric is spherically symmetric, any test particle within this spacetime has its energy and angular momentum conserved. Also because of this symmetry, it follows from equation (41) that the shadow of the Schwarzschild black hole is independent of the observation angle.

REFERENCES

[1] Bambi C, Freese K. 2009. Apparent shape of super-spinning black holes. Phys. Rev. D. 79:043002.

[2] Takahashi R. 2004. Shapes and Positions of Black Hole Shadows in Accretion Disks and Spin Parameters of Black Holes. Astrophys. J. 611:996.

[3] Cunningham C, Bardeen J. 1973. The Optical Appearance of a Star Orbiting an Extreme Kerr Black Hole. Astrophys. J. 183:237.

[4] Luminet J. 1979. Image of a spherical black hole with thin accretion disk. Astron. Astrophys. 75:228.

[5] The Event Horizon Telescope Collaboration. 2019. First M87 event horizon telescope results. I. The shadow of the supermassive black hole. Astrophys. J. Lett. 875:L1.

[6] The Event Horizon Telescope Collaboration. 2019. First M87 event horizon telescope results. II. Array and instrumentation. Astrophys. J. Lett. 875:L2.

[7] The Event Horizon Telescope Collaboration. 2019. First M87 event horizon telescope results. III. Data processing and calibration. Astrophys. J. Lett. 875:L3.

[8] The Event Horizon Telescope Collaboration. 2019. First M87 event horizon telescope results. IV. Imaging the central supermassive black hole. Astrophys. J. Lett. 875:L4.

[9] The Event Horizon Telescope Collaboration. 2019. First M87 event horizon telescope results. V. Physical origin of the asymmetric ring. Astrophys. J. Lett. 875:L5.

[10] The Event Horizon Telescope Collaboration. 2019. First M87 event horizon telescope results. VI. The shadow and mass of the central black hole. Astrophys. J. Lett. 875:L6.

[11] Synge JL. 1966. The escape of photons from gravitationally intense stars. MNRAS. 131:463.

[12] Chandrasekhar S. 1992. The Mathematical Theory of Black Holes. New York: Oxford University Press.

[13] Bardeen JM. 1973. Timelike and null geodesics in the Kerr metric. Les Astres Occlus.

**Author for Correspondence:* aulaslivres@gmail.com

[14] Event Horizon Telescope. [cited 2025 May 28]. Available from: https://eventhorizontelescope.org

[15] Mishra AK, Chakraborty S, Sarkar S. 2019. Understanding photon sphere and black hole shadow in dynamically evolving spacetimes. Phys. Rev. D. 99:10.

[16] Hobson MP, Efstathiou GP, Lasenby AN. 2006. General relativity: an introduction for physicists. Cambridge: Cambridge University Press.

[17] Ryder L. 2009. Introduction to General Relativity. Cambridge: Cambridge University Press.

[18] Villegas F. 2021. Ecuaciones de Friedmann en las teorías de gravedad modificada f(R). Rev. Invest. Física. 24(3). doi: https://doi.org/10.15381/rif.v24i3.21161

[19] Schwarzschild K. 1999. On the gravitational field of a mass point according to Einstein's theory. arXiv:physics/9905030. Translation and foreword by S. Antoci and A. Loinger.

[20] Einstein A. 1915. Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin. 844.

[21] Vazquez SE, Esteban EP. 2003. Strong Field Gravitational Lensing by a Kerr Black Hole. Nuovo Cim.B. 119.

[22] Bray I. 1986. Kerr black hole as a gravitational lens. Phys. Rev. D. 34:367.

[23] Cosino B. 2018. Introduction to General Relativity: A course for Undergraduate Student of Physics. Springer.

[24] de la Torre L. 2008. Elementos de relatividad. Medellín: Universidad de Antioquia.

[25] Goldstein H, Poole C, Safko J. 2002. Classical Mechanics. New York: Addison-Wesley.

[26] Carter B. 1968. Global Structure of the Kerr Family of Gravitational Fields. Phys. Rev. 174.

Villegas F. 2022. Shadow of the Schwarzschild black hole. Rev. Invest. Física. 25(3). doi: https://doi.org/10.15381/rif.v25i3.23592