RESEARCH PAPER

Exploring Structural Aspects of Complementary Triple Connected Total Domination Number in Product of Graphs

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Abstract: A Dominating set $S \subseteq V$ of a graph G is said to be Complementary triple connected total dominating set (CTCTD-set) if $\langle S \rangle$ has no isolated vertices and $\langle V - S \rangle$ is triple connected. The complementary triple connected total domination number CTCTD(G) is the minimum cardinality among all such sets in G. This article focuses on evaluating the CTCTD- number for Cartesian product of paths and cycles. Our investigation highlights how specific graph structures shape the CTCTD- number and contributes meaningful findings to the study of domination in graph products.

***Keywords**:* Total Domination, Triple connected Domination, Cartesian Product.

How to cite this article: Kaladevi M, Mahadevan G, Sivagnanam C. Exploring Structural Aspects of Complementary Triple Connected Total Domination Number in Product of Graphs. Int J Drug Deliv Technol. 2026;16(61s):985-1002. DOI: 10.25258/ijddt.16.61s.109
Source of support: Nil.
Conflict of interest: None

Introduction

The concept of complementary triple connected total dominating set (CTCTD- set) was introduced by Dr. G. Mahadevan as an enhancement of classical domination concepts in graph theory, blending structural connectivity with domination properties. A Dominating set $S \subseteq V$ of a graph G is said to be Complementary triple connected total dominating set (CTCTD-set) if $\langle S \rangle$ has no isolated vertices and $\langle V - S \rangle$ is triple connected, that is, it remains connected after the removal of any two vertices. The complementary triple connected total dominating number, denoted by CTCTD(G), is defined as the minimum cardinality of such a set S in G This paper investigates the CTCTD-number for Cartesian product graphs formed from paths and cycles.The Cartesian product of graphs G and H is the graph with vertex set $V(G) \times V(H)$ where (g_1, h_1) is adjacent to (g_2, h_2) iff either $g_1 = g_2$ and $h_1 h_2 \in (H)$, or $h_1 = h_2$ and $g_1 g_2 \in E(G)$. The study examines how the structure of these graph products influences their CTCTD-numbers and contributes to a deeper understanding of domination in such graphs.

1 Complementary Triple connected Total Domination Number of Cartesion Product of Path

Here, the exact CTCTD - number is calculated for all considered Cartesian Product of Paths, rather than giving only their bounds.

Definition 1.1
A Dominating set $S \subseteq V$ of a graph G is said to be Complementary triple connected total dominating set (CTCTD-set) if $\langle S \rangle$ has no isolated vertices and $\langle V - S \rangle$ is triple connected

Theorem 1.1 *If* $x \equiv 0 \ (mod\ 6)$ *and* $x \leq y$

$$\gamma_{CTCTD}(P_x \times P_y) = \begin{cases} \left(\frac{y}{2}\right)\left(\frac{2x}{3}+1\right) & if\ y \equiv 0\ (mod\ 4) \\ 2\lceil\frac{y}{4}\rceil\left(\frac{2x}{3}+1\right) & if\ y \equiv 3\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor+1\right)\left(\frac{2x}{3}+1\right) & otherwise \end{cases}$$

Proof :

Let $V(P_x \times P_y) = \{u_{mn}: 1 \leq m \leq x\,; 1 \leq n \leq y\}$, and let $E(P_x \times P_y) = \{u_{mn}u_{(m+1)n}: 1 \leq m \leq x-1;\ 1 \leq n \leq y\} \cup \{u_{mn}u_{m(n+1)}: 1 \leq m \leq x;\ \ 1 \leq n \leq y-1\}$.

Let $A_1 = \{u_{mn} : m \equiv 4\ or\ 5\ (mod\ 6);\ n \equiv 0\ or\ 1\ (mod\ 4)\} \cup \{u_{mn} : m \equiv 1\ or\ 2\ (mod\ 6);\ n \equiv 2\ or\ 3\ (mod\ 4)\} \cup \{u_{xn} : n \equiv 0\ or\ 1\ (mod\ 4)\}$.

$A_2 = \{u_{m(n-1)} : m \equiv 1\ or\ 2\ (mod\ 6)\}$.

$A_3 = \{u_{m(n-1)} : m \equiv 4\ or\ 5\ (mod\ 6)\}$.

Assume

$$A = \begin{cases} A_1 & if\ y \equiv 0\ or\ 2\ (mod\ 4) \\ A_1 \cup\ A_2 & if\ y \equiv 1\ (mod\ 4) \\ A_1 \cup\ A_3 & if\ y \equiv 3\ (mod\ 4) \end{cases}$$

Then A is a CTCTD-set of $P_x \times P_y$ and hence

$$\gamma_{CTCTD}(P_x \times P_y) \leq |A| = \begin{cases} \left(\frac{y}{2}\right)\left(\frac{2x}{3}+1\right) & if\ y \equiv 0\ (mod\ 4) \\ 2\lceil\frac{y}{4}\rceil\left(\frac{2x}{3}+1\right) & if\ y \equiv 3\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor+1\right)\left(\frac{2x}{3}+1\right) & otherwise \end{cases}$$

Let A' be a CTCTD-set of $P_x \times P_y$. Since $\langle V - \mathcal{D} \rangle$ is disconnected for any dominating set $\mathcal{D}$ of cardinality less than or equal to p, where

$$p = \begin{cases} \left(\frac{y}{2}\right)\left(\frac{2x}{3}+1\right)-1 & if\ y \equiv 0\ (mod\ 4) \\ 2\lceil\frac{y}{4}\rceil\left(\frac{2x}{3}+1\right)-1 & if\ y \equiv 3\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor+1\right)\left(\frac{2x}{3}+1\right)-1 & otherwise \end{cases}$$

we have

$$|A'| \geq p+1 = \begin{cases} \left(\frac{y}{2}\right)\left(\frac{2x}{3}+1\right) & if\ y \equiv 0\ (mod\ 4) \\ 2\lceil\frac{y}{4}\rceil\left(\frac{2x}{3}+1\right) & if\ y \equiv 3\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor+1\right)\left(\frac{2x}{3}+1\right) & otherwise \end{cases}$$

Hence the result follows.

Example 1.1

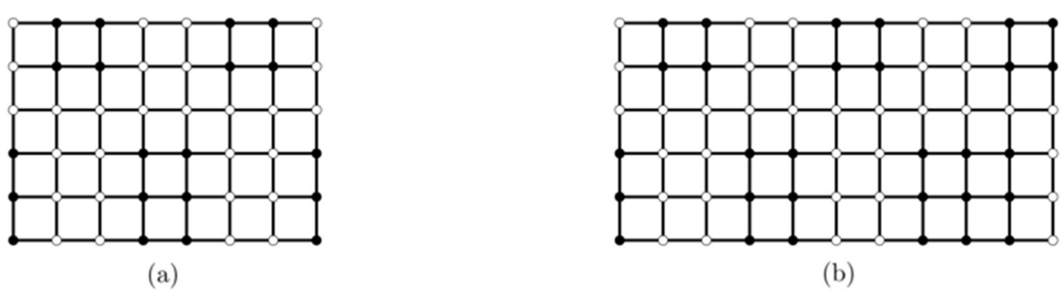


Figure 1: $(a)\ P_6 \times P_8$, $(b)\ P_6 \times P_{11}$

Here the set of darkened vertices denoted a CTCTD - set

In Figure 1 (a), $\gamma_{CTCTD}\ (P_6 \times P_8)$=$2\left(\frac{8}{4}\right)\left(4\left(\frac{6}{6}\right)+1\right)$ =20.

In Figure 1 (b), $\gamma_{CTCTD}\ (P_6 \times P_{11})$=$2\lceil\frac{11}{4}\rceil\left(4\left(\frac{6}{6}\right)+1\right)$ =30.

Theorem 1.2 *If* $x \equiv 1\ (mod\ 6)$ *and* $x \leq y$

$$\gamma_{CTCTD}(P_x \times P_y) = \begin{cases} y\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) & if\ y \equiv 0\ (mod\ 4) \\ 4\lceil\frac{y}{4}\rceil\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) & if\ y \equiv 3\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right)\right) & otherwise \end{cases}$$

Proof :

Let $V(P_x \times P_y) = \{u_{mn}: 1 \leq m \leq x\ ; 1 \leq n \leq y\}$, and let

$E(P_x \times P_y) = \{u_{mn}u_{(m+1)n}: 1 \leq m \leq x-1\ ; 1 \leq n \leq y\} \cup \{u_{mn}u_{m(n+1)}: 1 \leq m \leq x\ ;$
$1 \leq n \leq y-1\}$.

Let $A_1 = \{u_{mn}\ :\ m\ \equiv 4\ or\ 5\ (mod\ 6);\ n\ \equiv 0\ or\ 1\ (mod\ 4)\} \cup \{u_{mn}\ :\ m\ \equiv$
$1\ or\ 2\ (mod\ 6);\ n\ \equiv 2\ or\ 3\ (mod\ 4); 1 \leq x \leq x-1\} \cup$
$\{u_{xn}, u_{(x-1)n}\ :\ n\ \equiv 0\ or\ 1\ (mod\ 4)\}$.

$A_2 = \{u_{m(n-1)}\ :\ m\ \equiv 1\ or\ 2\ (mod\ 6)\}$.

$A_3 = \{u_{m(n-1)}\ :\ m\ \equiv 4\ or\ 5\ (mod\ 6)\}$.

Assume

$$A = \begin{cases} A_1 & if\ y\ \equiv 0\ or\ 2\ (mod\ 4) \\ A_1 \cup\ A_2 & if\ y\ \equiv 1\ (mod\ 4) \\ A_1 \cup\ A_3 & if\ y\ \equiv 3\ (mod\ 4) \end{cases}$$

Then A is a CTCTD-set of $P_x \times P_y$ and hence

$$\gamma_{CTCTD}\ (P_x \times P_y) \leq |A| = \begin{cases} y\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) & if\ y \equiv 0\ (mod\ 4) \\ 4\lceil\frac{y}{4}\rceil\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) & if\ y \equiv 3\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right)\right) & otherwise \end{cases}$$

Let A' be a CTCTD-set of $P_x \times P_y$. Since $\langle V - \mathcal{D}\rangle$ is disconnected for any dominating set $\mathcal{D}$ of cardinality less than or equal to p, where

$$p = \begin{cases} y\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) - 1 & if\ y \equiv 0\ (mod\ 4) \\ 4\lceil\frac{y}{4}\rceil\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) - 1 & if\ y \equiv 3\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right)\right) - 1 & otherwise \end{cases}$$

we have

$$|A'| \geq p + 1 = \begin{cases} y\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) & if\ y \equiv 0\ (mod\ 4) \\ 4\lceil\frac{y}{4}\rceil\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) & if\ y \equiv 3\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right)\right) & otherwise \end{cases}$$

Hence the result follows.

Example 1.2

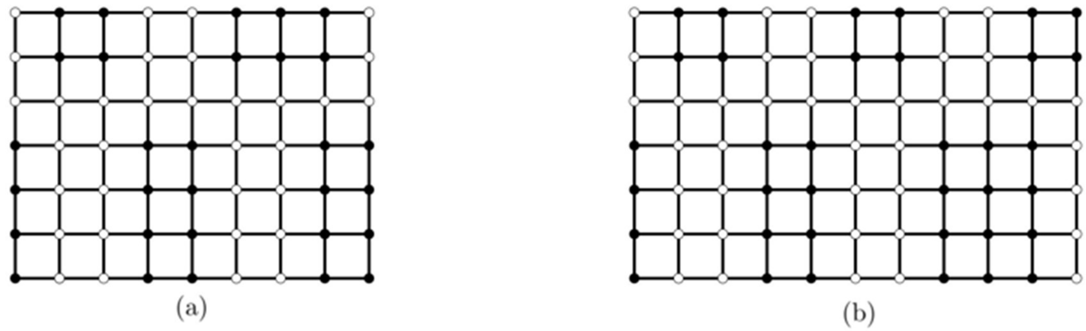


Figure 2: $(a)\ P_7 \times P_9$, $(b)\ P_7 \times P_{11}$

Here the set of darkened vertices denoted a CTCTD - set

In Figure 2 (a), $\gamma_{CTCTD}(P_7 \times P_{10})$=$\left(2\lfloor\frac{10}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{7}{6}\rceil + \lfloor\frac{7}{6}\rfloor\right)\right)$ =30.

In Figure 2 (b),$\gamma_{CTCTD}(P_7 \times P_{11})$=$4\lceil\frac{11}{4}\rceil\left(\lceil\frac{7}{6}\rceil + \lfloor\frac{7}{6}\rfloor\right)$ =36.

Theorem 1.3 *If* $x \equiv 2\ (mod\ 6)$ *and* $x \le y$

$$\gamma_{CTCTD}(P_x \times P_y) = \begin{cases} \frac{y}{2}\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y \equiv 0\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y \equiv 1\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right)\right) & if\ y \equiv 2\ (mod\ 4) \\ 4\lceil\frac{y}{4}\rceil\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) & if\ y \equiv 3\ (mod\ 4) \end{cases}$$

Proof :

Let $V(P_x \times P_y) = \{u_{mn}: 1 \le m \le x\ ; 1 \le n \le y\}$, and let

$$E(P_x \times P_y) = \{u_{mn}u_{(m+1)n}: 1 \le m \le x-1\ 1 \le n \le y\} \cup \{u_{mn}u_{m(n+1)}: 1 \le m \le x; 1 \le n \le y-1\}.$$

Let $A_1 = \{u_{mn} : m \equiv 4\ or\ 5\ (mod\ 6), n \equiv 0\ or\ 1\ (mod\ 4)\}$

$A_2 = \{u_{mn} : m \equiv 1\ or\ 2\ (mod\ 6), n \equiv 2\ or\ 3\ (mod\ 4), 1 \le x \le x-3\}$

$A_3 = \{u_{mn} : m \equiv 1\ or\ 2\ (mod\ 6), n \equiv 2\ or\ 3\ (mod\ 4)\}$

$A_4 = \{u_{(x-2)n}, u_{(x-1)n}, u_{xn} : n \equiv 0\ or\ 1\ (mod\ 4)\}$.

$A_5 = \{u_{m(y-1)} : m \equiv 1\ or\ 2\ (mod\ 6)\}$.

$A_6 = \{u_{m(y-1)} : m \equiv 4\ or\ 5\ (mod\ 6)\}$.

$A_7 = \{u_{my} : m \equiv 1\ or\ 2\ (mod\ 6)\}$.

$A_8 = \{u_{my} : m \equiv 4\ or\ 5\ (mod\ 6)\}$.

Assume

$$A = \begin{cases} A_1 \cup A_2 \cup A_4 & if\ y \equiv 0\ (mod\ 4) \\ A_1 \cup A_2 \cup A_4 \cup A_5 - A_7 & if\ y \equiv 1\ (mod\ 4) \\ A_1 \cup A_3 & if\ y \equiv 2\ (mod\ 4) \\ A_1 \cup A_3 \cup A_6 \cup A_5 - A_8 & if\ y \equiv 3\ (mod\ 4) \end{cases}$$

Then A is a CTCTD-set of $P_x \times P_y$ and hence

$$\gamma_{CTCTD}(P_x \times P_y) \leq |A| = \begin{cases} \frac{y}{2}\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y\ \equiv 0\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y\ \equiv 1\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right)\right) & if\ y\ \equiv 2\ (mod\ 4) \\ 4\lceil\frac{y}{4}\rceil\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) & if\ y\ \equiv 3\ (mod\ 4) \end{cases}$$

Let A' be a CTCTD-set of $P_x \times P_y$. Since $\langle V - \mathcal{D}\rangle$ is disconnected for any dominating set $\mathcal{D}$ of cardinality less than or equal to p, where

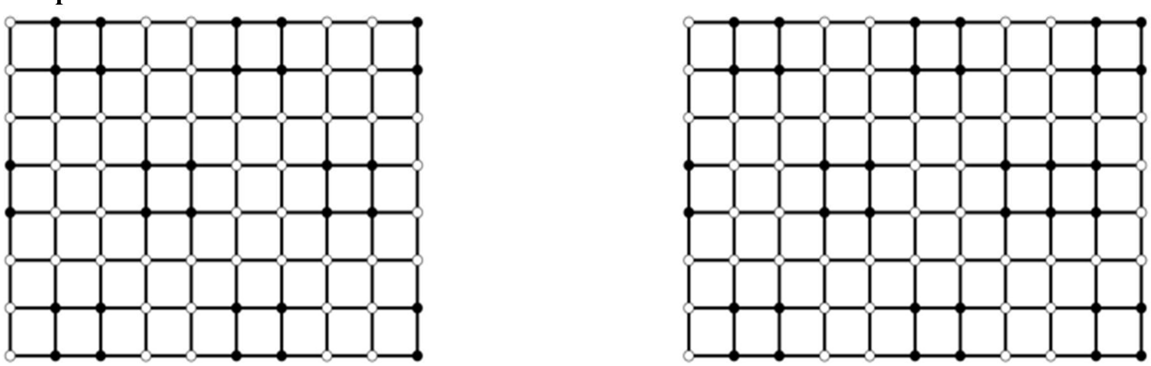
$$p = \begin{cases} \frac{y}{2}\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) - 1 & if\ y\ \equiv 0\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6} + \lfloor\frac{x}{6}\rfloor\right) + 1\right) - 1 & if\ y\ \equiv 1\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right)\right) - 1 & if\ y\ \equiv 2\ (mod\ 4) \\ 4\lceil\frac{y}{4}\rceil\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) - 1 & if\ y\ \equiv 3\ (mod\ 4) \end{cases}$$

we have

$$|A'| \geq p + 1 = \begin{cases} \frac{y}{2}\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y\ \equiv 0\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y\ \equiv 1\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right)\right) & if\ y\ \equiv 2\ (mod\ 4) \\ 4\lceil\frac{y}{4}\rceil\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) & if\ y\ \equiv 3\ (mod\ 4) \end{cases}$$

Hence the result follows.

Example 1.3



(a) (b)

Figure 3: $(a)\ P_8 \times P_{10}$, $(b)\ P_8 \times P_{11}$

Here the set of darkened vertices denoted a CTCTD - set

In Figure 3 (a), γ_{CTCTD} $(P_8 \times P_{10})$=$\left(2\lfloor\frac{10}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{8}{6}\rceil + \lfloor\frac{8}{6}\rfloor\right)\right)$ =30.

In Figure 3 (b), γ_{CTCTD} $(P_8 \times P_{11})$=$4\lceil\frac{y}{4}\rceil\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right)$ =36.

Theorem 1.4 *If* $x\ \equiv 3\ (mod\ 6)$ *and* $x \leq y$

$$\gamma_{CTCTD}(P_x \times P_y) = \begin{cases} y\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor + 1\right) & if\ y \equiv 0\ (mod\ 4) \\ 2\lceil\frac{y}{4}\rceil\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y \equiv 3\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(\left(2\lceil\frac{x}{6}\rceil + 1\right) + 2\lfloor\frac{x}{6}\rfloor\right) & otherwise \end{cases}$$

Proof :

Let $V(P_x \times P_y) = \{u_{mn}: 1 \leq m \leq x;\ 1 \leq n \leq y\}$, and let
$E(P_x \times P_y) = \{u_{mn}u_{(m+1)n}: 1 \leq m \leq x-1\ ; 1 \leq n \leq y\} \cup \{u_{mn}u_{m(n+1)}: 1 \leq m \leq x\ ; 1 \leq n \leq y-1\}$.

Let $A_1 = \{u_{mn} :\ m \equiv 4\ or\ 5\ (mod\ 6);\ n \equiv 0\ or\ 1\ (mod\ 4)\}$
$A_2 = \{u_{mn} :\ m \equiv 1\ or\ 2\ (mod\ 6);\ n \equiv 2\ or\ 3\ (mod\ 4); 1 \leq x \leq x-4\}$
$A_3 = \{u_{mn} :\ m \equiv 1\ or\ 2\ (mod\ 6);\ n \equiv 2\ or\ 3\ (mod\ 4)\}$
$A_4 = \{u_{(x-3)n}, u_{(x-2)n}, u_{(x-1)n}, u_{xn} :\ n \equiv 0\ or\ 1\ (mod\ 4)\}$.
$A_5 = \{u_{m(y-1)} :\ m \equiv 1\ or\ 2\ (mod\ 6)\}$.
$A_6 = \{u_{xy} :\ n \equiv 2\ or\ 3\ (mod\ 4)\}$.
$A_7 = \{u_{m(y-1)} :\ m \equiv 4\ or\ 5\ (mod\ 6)\}$.

Assume

$$A = \begin{cases} A_1 \cup\ A_2 \cup\ A_4 & if\ y \equiv 0\ (mod\ 4) \\ A_1 \cup\ A_2 \cup\ A_4 \cup\ A_5 & if\ y \equiv 1\ (mod\ 4) \\ A_1 \cup\ A_3 \cup\ A_6 & if\ y \equiv 2\ (mod\ 4) \\ A_1 \cup\ A_3 \cup\ A_6 \cup\ A_7 & if\ y \equiv 3\ (mod\ 4) \end{cases}$$

Then A is a CTCTD-set of $P_x \times P_y$ and hence

$$\gamma_{CTCTD}(P_x \times P_y) \leq |A| = \begin{cases} y\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor + 1\right) & if\ y \equiv 0\ (mod\ 4) \\ 2\lceil\frac{y}{4}\rceil\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y \equiv 3\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(\left(2\lceil\frac{x}{6}\rceil + 1\right) + 2\lfloor\frac{x}{6}\rfloor\right) & otherwise \end{cases}$$

Let A' be a CTCTD-set of $P_x \times P_y$. Since $\langle V - \mathcal{D} \rangle$ is disconnected for any dominating set $\mathcal{D}$ of cardinality less than or equal to p, where

$$p = \begin{cases} y\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor + 1\right) - 1 & if\ y \equiv 0\ (mod\ 4) \\ 2\lceil\frac{y}{4}\rceil\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) - 1 & if\ y \equiv 3\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(\left(2\lceil\frac{x}{6}\rceil + 1\right) + 2\lfloor\frac{x}{6}\rfloor\right) - 1 & otherwise \end{cases}$$

we have

$$|A'| \geq p + 1 = \begin{cases} y\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor + 1\right) & if\ y \equiv 0\ (mod\ 4) \\ 2\lceil\frac{y}{4}\rceil\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y \equiv 3\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(\left(2\lceil\frac{x}{6}\rceil + 1\right) + 2\lfloor\frac{x}{6}\rfloor\right) & otherwise \end{cases}$$

Hence the result follows.

Example 1.4

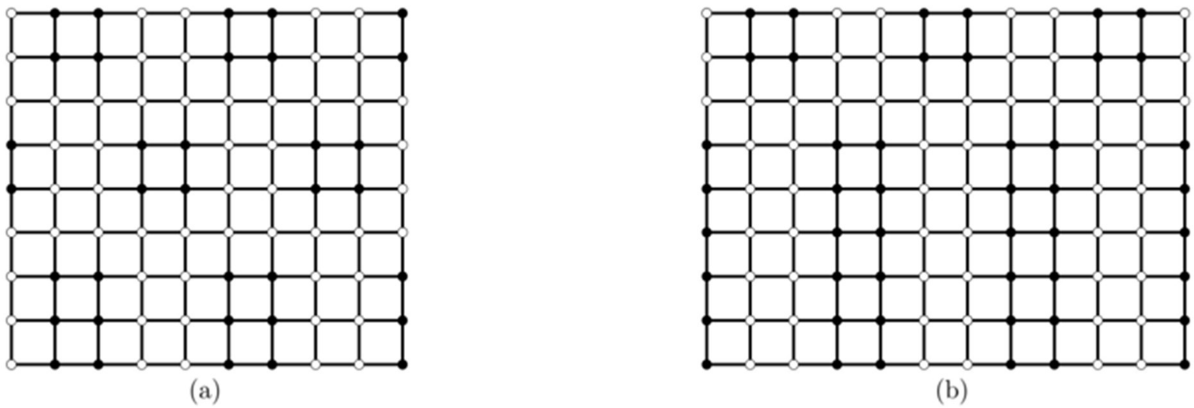


Figure 4: $(a)\ P_9 \times P_{10}$, $(b)\ P_9 \times P_{12}$

Here the set of darkened vertices denoted a CTCTD - set

In Figure 4 (a), $\gamma_{CTCTD}\ (P_9 \times P_{10}) = \left(2\lfloor\frac{10}{4}\rfloor + 1\right)\left(\left(2\lceil\frac{9}{6}\rceil + 1\right) + 2\lfloor\frac{9}{6}\rfloor\right)$ =35.

In Figure 4 (b), $\gamma_{CTCTD}\ (P_9 \times P_{12}) = 12\left(\lceil\frac{9}{6}\rceil + \lfloor\frac{9}{6}\rfloor + 1\right)$ =48.

Theorem 1.5 *If* $x \equiv 4\ (mod\ 6)$ *and* $x \le y$

$$\gamma_{CTCTD}(P_x \times P_y) = \begin{cases} \frac{y}{2}\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y \equiv 0\ (mod\ 4) \\ 2\lceil\frac{y}{4}\rceil\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y \equiv 3\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & otherwise \end{cases}$$

Proof :

Let $V(P_x \times P_y) = \{u_{mn} : 1 \le m \le x\ ; 1 \le n \le y\}$, and let

$E(P_x \times P_y) = \{u_{mn}u_{(m+1)n} : 1 \le m \le x-1;\ 1 \le n \le y\} \cup \{u_{mn}u_{m(n+1)} : 1 \le m \le x;\ 1 \le n \le y-1\}$.

Let

$A_1 = \{\{u_{mn} :\ m \equiv 4\ or\ 5\ (mod\ 6);\ n \equiv 0\ or\ 1\ (mod\ 4)\} \cup \{u_{mn} :\ m \equiv 1\ or\ 2\ (mod\ 6);\ n \equiv 2\ or\ 3\ (mod\ 4)\} \cup \{u_{(x-1)n} :\ n \equiv 2\ or\ 3\ (mod\ 4)\} - \{u_{(x-2)n} :\ n \equiv 2\ or\ 3\ (mod\ 4)\}\}$.

$A_2 = \{u_{m(y-1)} :\ m \equiv 1\ or\ 2\ (mod\ 6)\}$.

$A_3 = \{u_{m(y-1)} :\ m \equiv 4\ or\ 5\ (mod\ 6)\}$.

$A_4 = \{u_{(x-1)n}, u_{(x-2)n}\}$.

$A_5 = \{u_{(x-1)}(y-1)\}$.

$A_6 = \{u_{x(y-1)}, u_{(x-1)(y-1)}\}$.

Assume

$$A = \begin{cases} A_1 & if\ y \equiv 0\ \ (mod\ 4) \\ A_1 \cup\ A_2 & if\ y \equiv 1\ (mod\ 4) \\ A_1 \cup\ A_4 - A_6 & if\ y \equiv 2\ (mod\ 4) \\ A_1 \cup\ A_3 \cup\ A_5 & if\ y \equiv 3\ (mod\ 4) \end{cases}$$

Then A is a CTCTD-set of $P_x \times P_y$ and hence

$$\gamma_{CTCTD}(P_x \times P_y) \leq |A| = \begin{cases} 2\left(\frac{y}{4}\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y\ \equiv 0\ (mod\ 4) \\ 2\lceil\frac{y}{4}\rceil\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y\ \equiv 3\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & otherwise \end{cases}$$

Let A' be a CTCTD-set of $P_x \times P_y$. Since $\langle V - \mathcal{D}\rangle$ is disconnected for any dominating set $\mathcal{D}$ of cardinality less than or equal to p, where

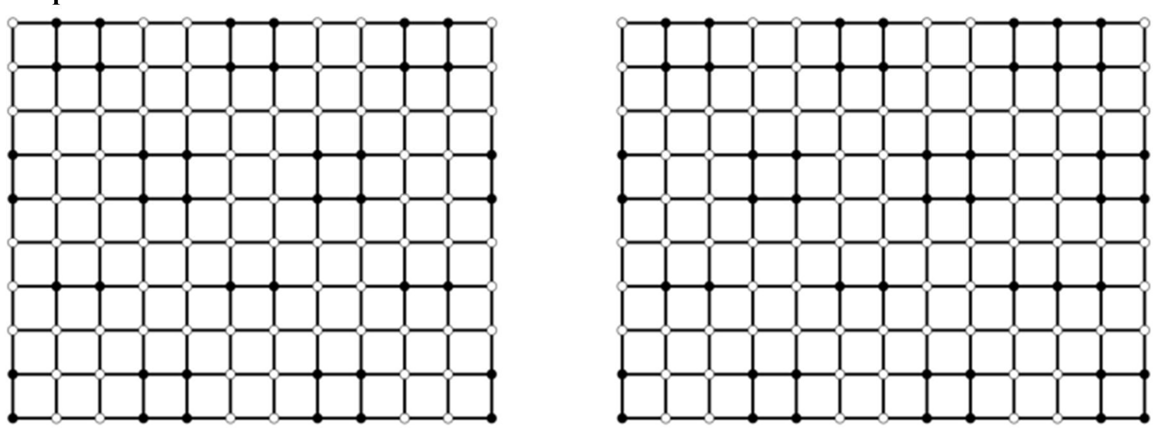
$$p = \begin{cases} 2\left(\frac{y}{4}\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) - 1 & if\ y\ \equiv 0\ (mod\ 4) \\ 2\lceil\frac{y}{4}\rceil\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) - 1 & if\ y\ \equiv 3\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & otherwise \end{cases}$$

we have

$$|A'| \geq p + 1 = \begin{cases} 2\left(\frac{y}{4}\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y\ \equiv 0\ (mod\ 4) \\ 2\lceil\frac{y}{4}\rceil\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y\ \equiv 3\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & otherwise \end{cases}$$

Hence the result follows.

Example 1.5



(a) (b)

Figure 5: $(a)\ P_{10} \times P_{12}$, $(b)\ P_{10} \times P_{13}$

Here the set of darkened vertices denoted a CTCTD - set

In Figure 5 (a), $\gamma_{CTCTD}(P_{10} \times P_{12})$=$2\left(\frac{12}{4}\right)\left(2\left(\lceil\frac{10}{6}\rceil + \lfloor\frac{10}{6}\rfloor\right) + 1\right)$ =42.

In Figure 5 (b), γ_{CTCTD} $(P_{10} \times P_{13})$=$\left(2\lfloor\frac{13}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{10}{6}\rceil + \lfloor\frac{10}{6}\rfloor\right) + 1\right)$ =49.

Theorem 1.6 *If* $x\ \equiv 5\ (mod\ 6)$ *and* $x \leq y$

$$\gamma_{CTCTD}(P_x \times P_y) = \begin{cases} 2y\lceil\frac{x}{6}\rceil & if\ y\ \equiv 0\ (mod\ 4) \\ 8\lceil\frac{x}{6}\rceil\lceil\frac{y}{4}\rceil & if\ y\ \equiv 3\ (mod\ 4) \\ 4\lceil\frac{x}{6}\rceil\left(2\lfloor\frac{y}{4}\rfloor + 1\right) & otherwise \end{cases}$$

Proof:

Let $V(P_x \times P_y) = \{u_{mn}: 1 \leq m \leq x\ ; 1 \leq n \leq y\}$, and let

$E(P_x \times P_y) = \{u_{mn}u_{(m+1)n}: 1 \leq m \leq x-1\, ; 1 \leq n \leq y\} \cup \{u_{mn}u_{m(n+1)}: 1 \leq m \leq x;\ 1 \leq n \leq y-1\}$.

Let $A_1 = \{\{u_{mn} : m \equiv 4\ or\ 5\ (mod\ 6);\ n \equiv 0\ or\ 1\ (mod\ 4)\} \cup \{u_{mn} : m \equiv 1\ or\ 2\ (mod\ 6);\ n \equiv 2\ or\ 3\ (mod\ 4)\}\}$.

$A_2 = \{u_{m(y-1)} : m \equiv 1\ or\ 2\ (mod\ 6)\}$.

$A_3 = \{u_{m(y-1)} : m \equiv 4\ or\ 5\ (mod\ 6)\}$.

Assume

$$A = \begin{cases} A_1 & if\ y \equiv 0\ or\ 2\ (mod\ 4) \\ A_1 \cup\ A_2 & if\ y \equiv 1\ (mod\ 4) \\ A_1 \cup\ A_3 & if\ y \equiv 3\ (mod\ 4) \end{cases}$$

Then A is a CTCTD-set of $P_x \times P_y$ and hence

$$\gamma_{CTCTD}(P_x \times P_y) \leq |A| = \begin{cases} 2y\lceil \frac{x}{6} \rceil & if\ y \equiv 0\ (mod\ 4) \\ 8\lceil \frac{x}{6} \rceil \lceil \frac{y}{4} \rceil & if\ y \equiv 3\ (mod\ 4) \\ 4\lceil \frac{x}{6} \rceil \left(2\lfloor \frac{y}{4} \rfloor + 1\right) & otherwise \end{cases}$$

Let A' be a CTCTD-set of $P_x \times P_y$. Since $\langle V - \mathcal{D} \rangle$ is disconnected for any dominating set $\mathcal{D}$ of cardinality less than or equal to p, where

$$p = \begin{cases} 2y\lceil \frac{x}{6} \rceil - 1 & if\ y \equiv 0\ (mod\ 4) \\ 8\lceil \frac{x}{6} \rceil \lceil \frac{y}{4} \rceil - 1 & if\ y \equiv 3\ (mod\ 4) \\ 4\lceil \frac{x}{6} \rceil \left(2\lfloor \frac{y}{4} \rfloor + 1\right) - 1 & otherwise \end{cases}$$

we have

$$|A'| \geq p + 1 = \begin{cases} 2y\lceil \frac{x}{6} \rceil & if\ y \equiv 0\ (mod\ 4) \\ 8\lceil \frac{x}{6} \rceil \lceil \frac{y}{4} \rceil & if\ y \equiv 3\ (mod\ 4) \\ 4\lceil \frac{x}{6} \rceil \left(2\lfloor \frac{y}{4} \rfloor + 1\right) & otherwise \end{cases}$$

Hence the result follows.

Example 1.6

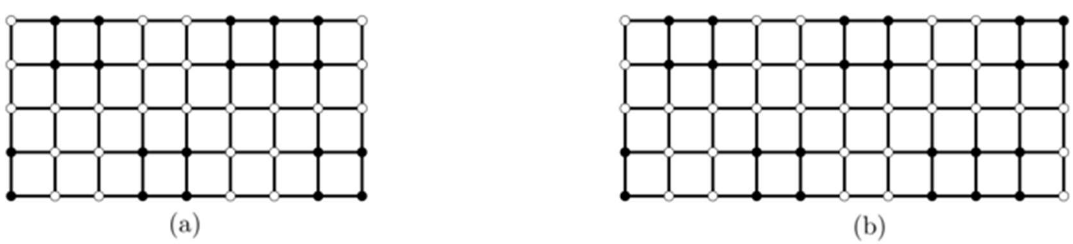


Figure 6: $(a)\ P_5 \times P_9$, $(b)\ P_5 \times P_{11}$

Here the set of darkened vertices denoted a CTCTD - set

In Figure 6 (a), $\gamma_{CTCTD}(P_5 \times P_9)$=$4\lceil \frac{5}{6} \rceil \left(2\lfloor \frac{9}{4} \rfloor + 1\right)$ =20.

In Figure 6 (b), $\gamma_{CTCTD}(P_5 \times P_{11})$=$8\lceil \frac{5}{6} \rceil \lceil \frac{11}{4} \rceil$ =24.

2 Complementary Triple connected Total Domination Number of Cartesion Product of Cycle

Theorem 2.1 *If* $x \equiv 0\ (mod\ 6)$ *and* $x \leq y$

$$\gamma_{CTCTD}(C_x \times C_y) = \begin{cases} y\left(\frac{x}{3}\right) & if\ y \equiv 0\ (mod\ 4) \\ \frac{2x}{3}\left(\lceil\frac{y}{4}\rceil + \lfloor\frac{y}{4}\rfloor\right) - 1 & if\ y \equiv 1\ (mod\ 4) \\ \frac{2x}{3}\left(2\lfloor\frac{y}{4}\rfloor + 1\right) & if\ y \equiv 2\ (mod\ 4) \\ \frac{x}{3}\left(2\left(\lceil\frac{y}{4}\rceil + \lfloor\frac{y}{4}\rfloor\right) + 1\right) & if\ y \equiv 3\ (mod\ 4) \end{cases}$$

Proof :

Let $V(C_x \times C_y) = \{u_{mn}: 1 \le m \le x\ ; 1 \le n \le y\}$, and let

$E(C_x \times C_y) = \{u_{mn}u_{(m+1)n}: 1 \le m \le x-1\ ; 1 \le n \le y\} \cup \{u_{mn}u_{m(n+1)}: 1 \le m \le x\ ; 1 \le n \le y-1\} \cup \{u_{1n}u_{xn}: 1 \le n \le y\} \cup \{u_{m1}u_{my}: 1 \le m \le x\}$.

Let $A_1 = \{\{u_{mn} :\ m \equiv 1\ or\ 2\ (mod\ 6);\ n \equiv 0\ or\ 1\ (mod\ 4)\} \cup \{u_{mn} :\ m \equiv 4\ or\ 5\ (mod\ 6);\ n \equiv 2\ or\ 3\ (mod\ 4)\}\}$.

$A_2 = \{u_{m(y-1)} :\ m \equiv 1\ or\ 2\ (mod\ 6)\}$.

$A_3 = \{u_{m(y-1)} :\ m \equiv 4\ or\ 5\ (mod\ 6)\}$.

Assume

$$A = \begin{cases} A_1 \cup A_2 & if\ y \equiv 1\ (mod\ 4) \\ A_1 & otherwise \end{cases}$$

Then A is a CTCTD-set of $C_x \times C_y$ and hence

$$\gamma_{CTCTD}(C_x \times C_y) \le |A| = \begin{cases} y\left(\frac{x}{3}\right) & if\ y \equiv 0\ (mod\ 4) \\ \frac{2x}{3}\left(\lceil\frac{y}{4}\rceil + \lfloor\frac{y}{4}\rfloor\right) - 1 & if\ y \equiv 1\ (mod\ 4) \\ \frac{2x}{3}\left(2\lfloor\frac{y}{4}\rfloor + 1\right) & if\ y \equiv 2\ (mod\ 4) \\ \frac{x}{3}\left(2\left(\lceil\frac{y}{4}\rceil + \lfloor\frac{y}{4}\rfloor\right) + 1\right) & if\ y \equiv 3\ (mod\ 4) \end{cases}$$

Let A' be a CTCTD-set of $C_x \times C_y$. Since $\langle V - \mathcal{D}\rangle$ is disconnected for any dominating set $\mathcal{D}$ of cardinality less than or equal to p, where

$$p = \begin{cases} y\left(\frac{x}{3}\right) - 1 & if\ y \equiv 0\ (mod\ 4) \\ \frac{2x}{3}\left(\lceil\frac{y}{4}\rceil + \lfloor\frac{y}{4}\rfloor\right) - 2 & if\ y \equiv 1\ (mod\ 4) \\ \frac{2x}{3}\left(2\lfloor\frac{y}{4}\rfloor + 1\right) - 1 & if\ y \equiv 2\ (mod\ 4) \\ \frac{x}{3}\left(2\left(\lceil\frac{y}{4}\rceil + \lfloor\frac{y}{4}\rfloor\right) + 1\right) - 1 & if\ y \equiv 3\ (mod\ 4) \end{cases}$$

we have

$$|A'| \ge p + 1 = \begin{cases} y\left(\frac{x}{3}\right) & if\ y \equiv 0\ (mod\ 4) \\ \frac{2x}{3}\left(\lceil\frac{y}{4}\rceil + \lfloor\frac{y}{4}\rfloor\right) - 1 & if\ y \equiv 1\ (mod\ 4) \\ \frac{2x}{3}\left(2\lfloor\frac{y}{4}\rfloor + 1\right) & if\ y \equiv 2\ (mod\ 4) \\ \frac{x}{3}\left(2\left(\lceil\frac{y}{4}\rceil + \lfloor\frac{y}{4}\rfloor\right) + 1\right) & if\ y \equiv 3\ (mod\ 4) \end{cases}$$

Hence the result follows.

Example 2.1

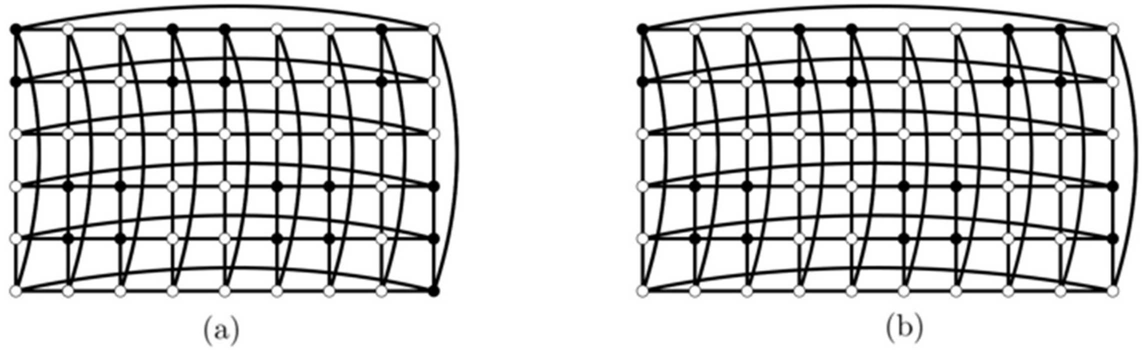


Figure 7: $(a)\ C_6 \times C_9$, $(b)\ C_6 \times C_{10}$

Here the set of darkened vertices denoted a CTCTD - set

In Figure 7 (a),$\gamma_{CTCTD}(C_6 \times C_9)=4\left(\frac{6}{6}\right)\left(\lceil\frac{9}{4}\rceil+\lfloor\frac{9}{4}\rfloor\right)-1$ =19.

In Figure 7 (b),$\gamma_{CTCTD}\ (C_6 \times C_{10})=4\left(\frac{6}{6}\right)\left(2\lfloor\frac{10}{4}\rfloor+1\right)$ =20.

Theorem 2.2 *If* $x \equiv 1\ (mod\ 6)$ *and* $x \leq y$

$$\gamma_{CTCTD}(C_x \times C_y) = \begin{cases} \frac{y}{2}\left(4\lfloor\frac{x}{6}\rfloor+1\right) & if\ y \equiv 0\ (mod\ 4) \\ 4\left(\lfloor\frac{x}{6}\rfloor\right)\left(\lceil\frac{y}{4}\rceil+\lfloor\frac{y}{4}\rfloor\right)+2\lfloor\frac{y}{4}\rfloor & if\ y \equiv 1\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor+1\right)\left(4\lfloor\frac{x}{6}\rfloor+1\right) & if\ y \equiv 2\ (mod\ 4) \\ 2\left(\lfloor\frac{x}{6}\rfloor\right)\left(2\left(\lceil\frac{y}{4}\rceil+\lfloor\frac{y}{4}\rfloor\right)+1\right)+2\lfloor\frac{y}{4}\rfloor & if\ y \equiv 3\ (mod\ 4) \end{cases}$$

Proof:

Let $V(C_x \times C_y) = \{u_{mn}: 1 \leq m \leq x\ ; 1 \leq n \leq y\}$, and let

$E(C_x \times C_y) = \{u_{mn}u_{(m+1)n}: 1 \leq m \leq x-1\ ; 1 \leq n \leq y\} \cup \{u_{mn}u_{m(n+1)}: 1 \leq m \leq x\ ; 1 \leq n \leq y-1\} \cup \{u_{1n}u_{xn}: 1 \leq n \leq y\} \cup \{u_{m1}u_{my}: 1 \leq m \leq x\}$.

Let $A_1 = \{\{u_{mn} : m \equiv 4\ or\ 5\ (mod\ 6);\ n \equiv 2\ or\ 3\ (mod\ 4)\}\}$.

$A_2 = \{u_{mn} : m \equiv 1\ or\ 2\ (mod\ 6);\ n \equiv 0\ or\ 1\ (mod\ 4); 1 \leq m \leq x-1\}$.

$A_3 = \{u_{mn} : m \equiv 1\ or\ 2\ (mod\ 6);\ n \equiv 0\ or\ 1\ (mod\ 4); 1 \leq m \leq x-1; 1 \leq n \leq y-1\}$.

$A_4 = \{\{u_{m(y-1)}, u_{my} : m \equiv 4\ or\ 5\ (mod\ 6)\} \cup \{u_{(x-1)y}, u_{xy}\} - \{u_{(x-3)(y-1)\}}, u_{(x-2)(y-1)}\}$.

Assume

$$A = \begin{cases} A_1 \cup A_3 \cup A_4 & if\ y \equiv 1\ (mod\ 4) \\ A_1 \cup A_2 & otherwise \end{cases}$$

Then A is a CTCTD-set of $C_x \times C_y$ and hence

$$\gamma_{CTCTD}(C_x \times C_y) \leq |A| = \begin{cases} \frac{y}{2}\left(4\lfloor\frac{x}{6}\rfloor+1\right) & if\ y \equiv 0\ (mod\ 4) \\ 4\left(\lfloor\frac{x}{6}\rfloor\right)\left(\lceil\frac{y}{4}\rceil+\lfloor\frac{y}{4}\rfloor\right)+2\lfloor\frac{y}{4}\rfloor & if\ y \equiv 1\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor+1\right)\left(4\lfloor\frac{x}{6}\rfloor+1\right) & if\ y \equiv 2\ (mod\ 4) \\ 2\left(\lfloor\frac{x}{6}\rfloor\right)\left(2\left(\lceil\frac{y}{4}\rceil+\lfloor\frac{y}{4}\rfloor\right)+1\right)+2\lfloor\frac{y}{4}\rfloor & if\ y \equiv 3\ (mod\ 4) \end{cases}$$

Let A' be a CTCTD-set of $C_x \times C_y$. Since $\langle V - \mathcal{D} \rangle$ is disconnected for any dominating set $\mathcal{D}$ of cardinality less than or equal to p, where

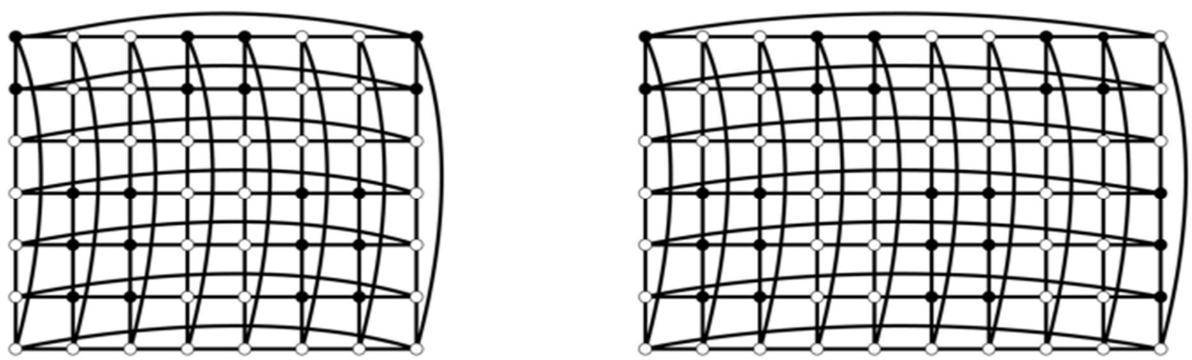
$$p = \begin{cases} \frac{y}{2}\left(4\lfloor\frac{x}{6}\rfloor + 1\right) - 1 & if\ y \equiv 0\ (mod\ 4) \\ 4\left(\lfloor\frac{x}{6}\rfloor\right)\left(\lceil\frac{y}{4}\rceil + \lfloor\frac{y}{4}\rfloor\right) + 2\lfloor\frac{y}{4}\rfloor - 1 & if\ y \equiv 1\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(4\lfloor\frac{x}{6}\rfloor + 1\right) - 1 & if\ y \equiv 2\ (mod\ 4) \\ 2\left(\lfloor\frac{x}{6}\rfloor\right)\left(2\left(\lceil\frac{y}{4}\rceil + \lfloor\frac{y}{4}\rfloor\right) + 1\right) + 2\lfloor\frac{y}{4}\rfloor - 1 & if\ y \equiv 3\ (mod\ 4) \end{cases}$$

we have

$$|A'| \geq p + 1 = \begin{cases} \frac{y}{2}\left(4\lfloor\frac{x}{6}\rfloor + 1\right) & if\ y \equiv 0\ (mod\ 4) \\ 4\left(\lfloor\frac{x}{6}\rfloor\right)\left(\lceil\frac{y}{4}\rceil + \lfloor\frac{y}{4}\rfloor\right) + 2\lfloor\frac{y}{4}\rfloor & if\ y \equiv 1\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(4\lfloor\frac{x}{6}\rfloor + 1\right) & if\ y \equiv 2\ (mod\ 4) \\ 2\left(\lfloor\frac{x}{6}\rfloor\right)\left(2\left(\lceil\frac{y}{4}\rceil + \lfloor\frac{y}{4}\rfloor\right) + 1\right) + 2\lfloor\frac{y}{4}\rfloor & if\ y \equiv 3\ (mod\ 4) \end{cases}$$

Hence the result follows.

Example 2.2



(a) (b)

Figure 8: $(a)\ C_7 \times C_8$, $(b)\ C_7 \times C_{10}$

Here the set of darkened vertices denoted a CTCTD - set

In Figure 8 (a), $\gamma_{CTCTD}(C_7 \times C_8)$=$2\left(\frac{8}{4}\right)\left(4\lfloor\frac{7}{6}\rfloor + 1\right)$ =20.

In Figure 8 (b),$\gamma_{CTCTD}(C_7 \times C_{10})$=$\left(2\lfloor\frac{10}{4}\rfloor + 1\right)\left(4\lfloor\frac{7}{6}\rfloor + 1\right)$ =25.

Theorem 2.3 *If* $x \equiv 2\ (mod\ 6)\ and\ x \leq y$

$$\gamma_{CTCTD}(C_x \times C_y) = \begin{cases} y\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) & if\ y \equiv 0\ (mod\ 4) \\ 4\lfloor\frac{y}{4}\rfloor\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 4\lfloor\frac{x}{6}\rfloor & if\ y \equiv 1\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right)\right) & if\ y \equiv 2\ (mod\ 4) \\ 2\lceil\frac{x}{6}\rceil\left(2\lfloor\frac{y}{4}\rfloor + 1\right) + 4\lfloor\frac{x}{6}\rfloor\lceil\frac{y}{4}\rceil & if\ y \equiv 3\ (mod\ 4) \end{cases}$$

Proof :

Let $V(C_x \times C_y) = \{u_{mn}: 1 \leq m \leq x\ ; 1 \leq n \leq y\}$, and let

$E(C_x \times C_y) = \{u_{mn}u_{(m+1)n}: 1 \leq m \leq x - 1\ ; 1 \leq n \leq y\} \cup \{u_{mn}u_{m(n+1)}: 1 \leq m \leq x\ ; 1 \leq n \leq y - 1\} \cup \{u_{1n}u_{xn}: 1 \leq n \leq y\} \cup \{u_{m1}u_{my}, 1 \leq m \leq x\}$.

Let

$A_1 = \{\{u_{mn} : m \equiv 1\ or\ 2\ (mod\ 6);\ n \equiv 0\ or\ 1\ (mod\ 4)\} \cup \{u_{mn} : m \equiv 4\ or\ 5\ (mod\ 6);\ n \equiv$

$2\ or\ 3\ (mod\ 4)\}\}$.
$A_2 = \{\{u_{m(y-1)} : m \equiv 4\ or\ 5\ (mod\ 6)\} - \{u_{1,y}, u_{2y}\}\}$.
Assume

$$A = \begin{cases} A_1 \cup A_2 & if\ y \equiv 1\ (mod\ 4) \\ A_1 & otherwise \end{cases}$$

Then A is a CTCTD-set of $C_x \times C_y$ and hence

$$\gamma_{CTCTD}(C_x \times C_y) \le |A| = \begin{cases} y\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) & if\ y \equiv 0\ (mod\ 4) \\ 4\lfloor\frac{y}{4}\rfloor\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 4\lfloor\frac{x}{6}\rfloor & if\ y \equiv 1\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right)\right) & if\ y \equiv 2\ (mod\ 4) \\ 2\lceil\frac{x}{6}\rceil\left(2\lfloor\frac{y}{4}\rfloor + 1\right) + 4\lfloor\frac{x}{6}\rfloor\lceil\frac{y}{4}\rceil & if\ y \equiv 3\ (mod\ 4) \end{cases}$$

Let A' be a CTCTD-set of $C_x \times C_y$. Since $\langle V - \mathcal{D} \rangle$ is disconnected for any dominating set $\mathcal{D}$ of cardinality less than or equal to p, where

$$p = \begin{cases} y\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) - 1 & if\ y \equiv 0\ (mod\ 4) \\ 4\lfloor\frac{y}{4}\rfloor\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 4\lfloor\frac{x}{6}\rfloor - 1 & if\ y \equiv 1\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right)\right) - 1 & if\ y \equiv 2\ (mod\ 4) \\ 2\lceil\frac{x}{6}\rceil\left(2\lfloor\frac{y}{4}\rfloor + 1\right) + 4\lfloor\frac{x}{6}\rfloor\lceil\frac{y}{4}\rceil - 1 & if\ y \equiv 3\ (mod\ 4) \end{cases}$$

we have

$$|A'| \ge p + 1 = \begin{cases} y\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) & if\ y \equiv 0\ (mod\ 4) \\ 4\lfloor\frac{y}{4}\rfloor\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 4\lfloor\frac{x}{6}\rfloor & if\ y \equiv 1\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right)\right) & if\ y \equiv 2\ (mod\ 4) \\ 2\lceil\frac{x}{6}\rceil\left(2\lfloor\frac{y}{4}\rfloor + 1\right) + 4\lfloor\frac{x}{6}\rfloor\lceil\frac{y}{4}\rceil & if\ y \equiv 3\ (mod\ 4) \end{cases}$$

Hence the result follows.

Example 2.3

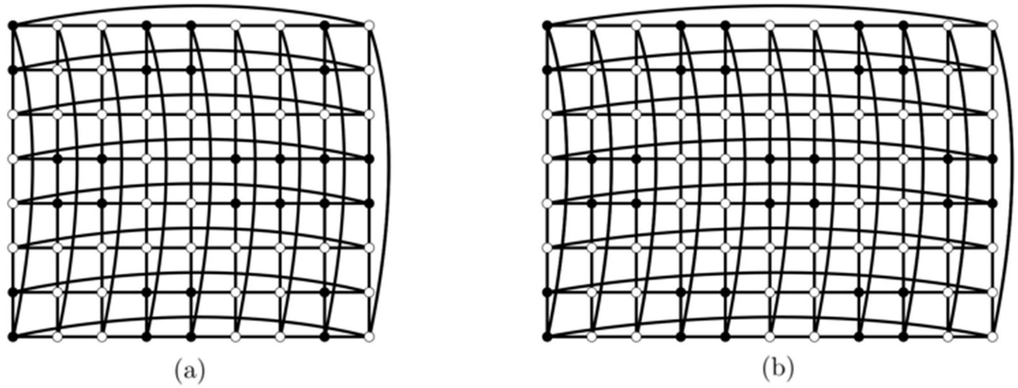


Figure 9: $(a)\ C_8 \times C_9$, $(b)\ C_8 \times C_{11}$

Here the set of darkened vertices denoted a CTCTD - set

In Figure 9 (a), $\gamma_{CTCTD}(C_8 \times C_9) = 4\lfloor\frac{9}{4}\rfloor\left(\lceil\frac{8}{6}\rceil + \lfloor\frac{8}{6}\rfloor\right) + 4\lfloor\frac{8}{6}\rfloor = 28$.

In Figure 9 (b), $\gamma_{CTCTD}(C_8 \times C_{11}) = 2\lceil\frac{8}{6}\rceil\left(2\lfloor\frac{11}{4}\rfloor + 1\right) + 4\lfloor\frac{8}{6}\rfloor\lceil\frac{11}{4}\rceil = 32$.

Theorem 2.4 *If* $x \equiv 3\ (mod\ 6)$ *and* $x \leq y$

$$\gamma_{CTCTD}(C_x \times C_y) = \begin{cases} \frac{y}{2}\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y \equiv 0\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\lceil\frac{x}{6}\rceil + 1\right) + 4\lfloor\frac{x}{6}\rfloor\lceil\frac{y}{4}\rceil & if\ y \equiv 3\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & otherwise \end{cases}$$

Proof :

Let $V(C_x \times C_y) = \{u_{mn}: 1 \leq m \leq x\ ; 1 \leq n \leq y\}$, and let

$E(C_x \times C_y) = \{u_{mn}u_{(m+1)n}: 1 \leq m \leq x-1\ ; 1 \leq n \leq y\} \cup \{u_{mn}u_{m(n+1)}: 1 \leq m \leq x\ ; 1 \leq n \leq y-1\} \cup \{u_{1n}u_{xn}: 1 \leq n \leq y\} \cup \{u_{m1}u_{my}: 1 \leq m \leq x\}$.

Let

$A_1 = \{\{u_{mn} : m \equiv 1\ or\ 2\ (mod\ 6);\ n \equiv 0\ or\ 1\ (mod\ 4)\} \cup \{u_{mn} : m \equiv 4\ or\ 5\ (mod\ 6);\ n \equiv 2\ or\ 3\ (mod\ 4)\} \cup \{u_{xn};\ n \equiv 0\ or\ 1\ (mod\ 4)\}\}$.

$A_2 = \{\{u_{m(y-1)} : m \equiv 4\ or\ 5\ (mod\ 6)\} - \{u_{1,y}, u_{2y}\}\}$.

Assume $$A = \begin{cases} A_1 \cup A_2 & if\ y \equiv 1\ (mod\ 4) \\ A_1 & otherwise \end{cases}$$

Then A is a CTCTD-set of $C_x \times C_y$ and hence

$$\gamma_{CTCTD}(C_x \times C_y) \leq |A| = \begin{cases} \frac{y}{2}\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y \equiv 0\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\lceil\frac{x}{6}\rceil + 1\right) + 4\lfloor\frac{x}{6}\rfloor\lceil\frac{y}{4}\rceil & if\ y \equiv 3\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & otherwise \end{cases}$$

Let A' be a CTCTD-set of $C_x \times C_y$. Since $\langle V - \mathcal{D}\rangle$ is disconnected for any dominating set $\mathcal{D}$ of cardinality less than or equal to p, where

$$p = \begin{cases} \frac{y}{2}\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) - 1 & if\ y \equiv 0\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\lceil\frac{x}{6}\rceil + 1\right) + 4\lfloor\frac{x}{6}\rfloor\lceil\frac{y}{4}\rceil - 1 & if\ y \equiv 3\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) - 1 & otherwise \end{cases}$$

we have $$|A'| \geq p + 1 = \begin{cases} \frac{y}{2}\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y \equiv 0\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\lceil\frac{x}{6}\rceil + 1\right) + 4\lfloor\frac{x}{6}\rfloor\lceil\frac{y}{4}\rceil & if\ y \equiv 3\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & otherwise \end{cases}$$

Hence the result follows.

Example 2.4

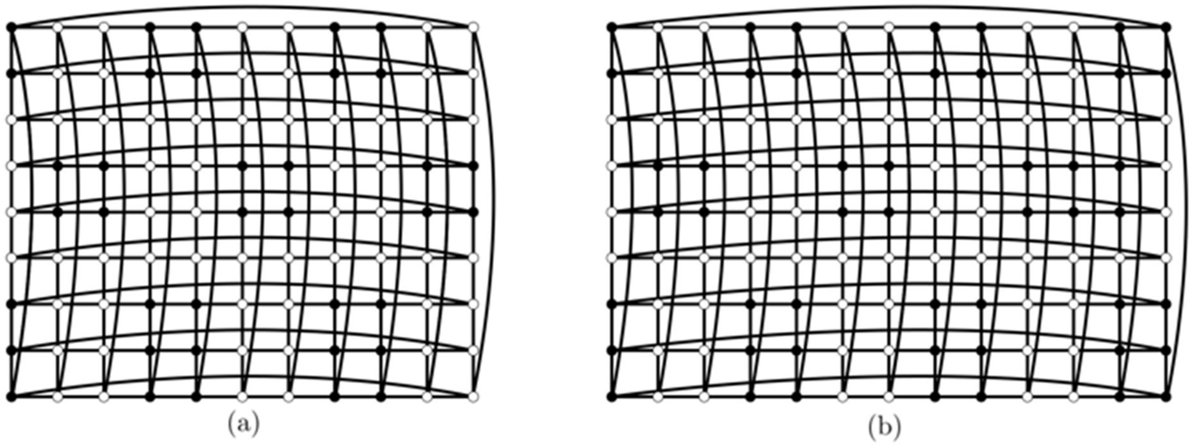


Figure 10: $(a)\ C_9 \times C_{11}$, $(b)\ C_9 \times C_{13}$

Here the set of darkened vertices denoted a CTCTD - set

In Figure 10 (a), $\gamma_{CTCTD}(C_9 \times C_{11})$=$\left(2\lfloor\frac{11}{4}\rfloor + 1\right)\left(2\lfloor\frac{9}{6}\rfloor + 1\right) + 4\lfloor\frac{9}{6}\rfloor\lceil\frac{13}{4}\rceil$ =37.

In Figure 10 (b), $\gamma_{CTCTD}(C_9 \times C_{13})$=$\left(2\lfloor\frac{13}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{10}{6}\rceil + \lfloor\frac{10}{6}\rfloor\right) + 1\right)$ =49.

Theorem 2.5 *If* $x \equiv 4\ (mod\ 6)$ *and* $x \leq y$

$$\gamma_{CTCTD}(C_x \times C_y) = \begin{cases} \frac{y}{2}\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y \equiv 0\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right)\right) + 2\lfloor\frac{y}{4}\rfloor & if\ y \equiv 1(mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y \equiv 2(mod\ 4) \\ 2\lceil\frac{y}{4}\rceil\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) - 2 & if\ y \equiv 3\ (mod\ 4) \end{cases}$$

Proof:

Let $V(C_x \times C_y) = \{u_{mn}: 1 \leq m \leq x\ ; 1 \leq n \leq y\}$, and let

$E(C_x \times C_y) = \{u_{mn}u_{(m+1)n}: 1 \leq m \leq x - 1\ ; 1 \leq n \leq y\} \cup \{u_{mn}u_{m(n+1)}: 1 \leq m \leq x\ ;\ 1 \leq n \leq y - 1\} \cup \{u_{1n}u_{xn}: 1 \leq n \leq y\} \cup \{u_{m1}u_{my}: 1 \leq m \leq x\}$.

Let $A_1 = \{\{u_{mn} : m \equiv 1\ or\ 2\ (mod\ 6);\ n \equiv 0\ or\ 1\ (mod\ 4)\} \cup \{u_{mn} : m \equiv 4\ or\ 5\ (mod\ 6);\ n \equiv 2\ or\ 3\ (mod\ 4)\}\}$.

$A_2 = \{\{u_{m(y-1)} : m \equiv 4\ or\ 5\ (mod\ 6)\} - \{u_{(x-1)(y-1)}\}\}$.

$A_3 = \{u_{m(y-1)} : m \equiv 1\ or\ 2\ (mod\ 6); 1 \leq m \leq x\}\}$.

$A_4 = \{u_{(x-1)y}\} - \{u_{(x-2)(y-1)}\}$.

Assume

$$A = \begin{cases} A_1 \cup\ A_2 & if\ y \equiv 1\ (mod\ 4) \\ A_1 & otherwise \end{cases}$$

Then A is a CTCTD-set of $C_x \times C_y$ and hence

$$\gamma_{CTCTD}(C_x \times C_y) \leq |A| = \begin{cases} \frac{y}{2}\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y \equiv 0\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right)\right) + 2\lfloor\frac{y}{4}\rfloor & if\ y \equiv 1(mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y \equiv 2(mod\ 4) \\ 2\lceil\frac{y}{4}\rceil\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) - 2 & if\ y \equiv 3\ (mod\ 4) \end{cases}$$

Let A' be a CTCTD-set of $C_x \times C_y$. Since $\langle V - \mathcal{D}\rangle$ is disconnected for any dominating set $\mathcal{D}$ of cardinality less than or equal to p, where

$$p=\begin{cases}\frac{y}{2}\left(2\left(\lceil\frac{x}{6}\rceil+\lfloor\frac{x}{6}\rfloor\right)+1\right)-1 & if\ y\equiv 0\ (mod\ 4)\\ \left(2\lfloor\frac{y}{4}\rfloor+1\right)\left(2\left(\lceil\frac{x}{6}\rceil+\lfloor\frac{x}{6}\rfloor\right)\right)+2\lfloor\frac{y}{4}\rfloor-1 & if\ y\equiv 1(mod\ 4)\\ \left(2\lfloor\frac{y}{4}\rfloor+1\right)\left(2\left(\lceil\frac{x}{6}\rceil+\lfloor\frac{x}{6}\rfloor\right)+1\right)-1 & if\ y\equiv 2(mod\ 4)\\ 2\lceil\frac{y}{4}\rceil\left(2\left(\lceil\frac{x}{6}\rceil+\lfloor\frac{x}{6}\rfloor\right)+1\right)-3 & if\ y\equiv 3\ (mod\ 4)\end{cases}$$

we have

$$|A'|\geq p+1=\begin{cases}\frac{y}{2}\left(2\left(\lceil\frac{x}{6}\rceil+\lfloor\frac{x}{6}\rfloor\right)+1\right) & if\ y\equiv 0\ (mod\ 4)\\ \left(2\lfloor\frac{y}{4}\rfloor+1\right)\left(2\left(\lceil\frac{x}{6}\rceil+\lfloor\frac{x}{6}\rfloor\right)\right)+2\lfloor\frac{y}{4}\rfloor & if\ y\equiv 1(mod\ 4)\\ \left(2\lfloor\frac{y}{4}\rfloor+1\right)\left(2\left(\lceil\frac{x}{6}\rceil+\lfloor\frac{x}{6}\rfloor\right)+1\right) & if\ y\equiv 2(mod\ 4)\\ 2\lceil\frac{y}{4}\rceil\left(2\left(\lceil\frac{x}{6}\rceil+\lfloor\frac{x}{6}\rfloor\right)+1\right)-2 & if\ y\equiv 3\ (mod\ 4)\end{cases}$$

Hence the result follows.

Example 2.5

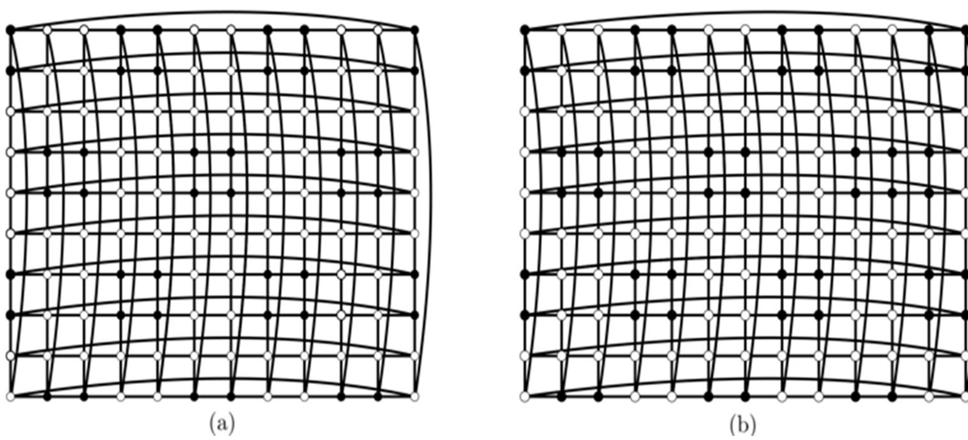


Figure 11: $(a)\ C_{10}\times C_{12}$, $(b)\ C_{10}\times C_{13}$

Here the darkened vertices denoted a CTCTD - set

In Figure 11 (a), $\gamma_{CTCTD}(C_{10}\times C_{12})$=$2\left(\frac{12}{4}\right)\left(2\left(\lceil\frac{10}{6}\rceil+\lfloor\frac{10}{6}\rfloor\right)+1\right)$ =42.

In Figure 11 (b), $\gamma_{CTCTD}(C_{10}\times C_{13})$=$\left(2\lfloor\frac{13}{4}\rfloor+1\right)\left(2\left(\lceil\frac{10}{6}\rceil+\lfloor\frac{10}{6}\rfloor\right)\right)+2\lfloor\frac{13}{4}\rfloor$ =48.

Theorem 2.6 *If* $x\equiv 5\ (mod\ 6)$ *and* $x\leq y$

$$\gamma_{CTCTD}(C_x\times C_y)=\begin{cases}\frac{y}{2}\left(2\left(\lceil\frac{x}{6}\rceil+\lfloor\frac{x}{6}\rfloor\right)+1\right) & if\ y\equiv 0\ (mod\ 4)\\ \left(2\lfloor\frac{y}{4}\rfloor+1\right)\left(2\left(\lceil\frac{x}{6}\rceil+\lfloor\frac{x}{6}\rfloor\right)+1\right) & if\ y\equiv 1(mod\ 4)\\ \left(2\lfloor\frac{y}{4}\rfloor+1\right)\left(2\left(\lceil\frac{x}{6}\rceil+\lfloor\frac{x}{6}\rfloor\right)+1\right)+1 & if\ y\equiv 2(mod\ 4)\\ 4\lceil\frac{x}{6}\rceil\lfloor\frac{y}{4}\rfloor+2\lceil\frac{y}{4}\rceil\left(2\lceil\frac{x}{6}\rceil+1\right) & if\ y\equiv 3\ (mod\ 4)\end{cases}$$

Proof :

Let $V(C_x\times C_y)=\{u_{mn}: 1\leq m\leq x\ ; 1\leq n\leq y\}$, and let

$E(C_x\times C_y)=\{u_{mn}u_{(m+1)n}: 1\leq m\leq x-1\ ; 1\leq n\leq y\}\cup\{u_{mn}u_{m(n+1)}: 1\leq m\leq$

$x\,; 1 \le n \le y-1\} \cup \{u_{1n}u_{xn}: 1 \le n \le y\} \cup \{u_{m1}u_{my}: 1 \le m \le x\}$.

Let

$A_1 = \{\{u_{mn} : m \equiv 1 \text{ or } 2 \ (mod\ 6);\ n \equiv 0 \text{ or } 1 \ (mod\ 4)\} \cup \{u_{mn} : m \equiv 4 \text{ or } 5 \ (mod\ 6);\ n \equiv 2 \text{ or } 3 \ (mod\ 4)\} - \{u_{xn}\,;\ n \equiv 2 \text{ or } 3 \ (mod\ 4)\}\}$.

$A_2 = \{\{u_{m(y-1)} : m \equiv 4 \text{ or } 5 \ (mod\ 6); 1 \le m \le x-3\} - \{u_{(x-2)y}, u_{(x-1)y}, u_{xy}\}\}$.

Assume

$$A = \begin{cases} A_1 & if\ y \equiv 0 \text{ or } 3\ (mod\ 4) \\ A_1 \cup A_2 & if\ y \equiv 1\ (mod\ 4) \\ A_1 \cup \{u_{xy}\} & if\ y \equiv 2\ (mod\ 4) \end{cases}$$

Then A is a CTCTD-set of $C_x \times C_y$ and hence

$$\gamma_{CTCTD}\,(C_x \times C_y) \le |A| = \begin{cases} \frac{y}{2}\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y \equiv 0\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y \equiv 1 (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) + 1 & if\ y \equiv 2 (mod\ 4) \\ 4\lceil\frac{x}{6}\rceil\lfloor\frac{y}{4}\rfloor + 2\lceil\frac{y}{4}\rceil\left(2\lfloor\frac{x}{6}\rfloor + 1\right) & if\ y \equiv 3\ (mod\ 4) \end{cases}$$

Let A' be a CTCTD-set of $C_x \times C_y$. Since $\langle V - \mathcal{D}\rangle$ is disconnected for any dominating set $\mathcal{D}$ of cardinality less than or equal to p, where

$$p = \begin{cases} \frac{y}{2}\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) - 1 & if\ y \equiv 0\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) - 1 & if\ y \equiv 1 (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y \equiv 2 (mod\ 4) \\ 4\lceil\frac{x}{6}\rceil\lfloor\frac{y}{4}\rfloor + 2\lceil\frac{y}{4}\rceil\left(2\lfloor\frac{x}{6}\rfloor + 1\right) - 1 & if\ y \equiv 3\ (mod\ 4) \end{cases}$$

we have

$$|A'| \ge p + 1 = \begin{cases} \frac{y}{2}\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y \equiv 0\ (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) & if\ y \equiv 1 (mod\ 4) \\ \left(2\lfloor\frac{y}{4}\rfloor + 1\right)\left(2\left(\lceil\frac{x}{6}\rceil + \lfloor\frac{x}{6}\rfloor\right) + 1\right) + 1 & if\ y \equiv 2 (mod\ 4) \\ 2\lceil\frac{x}{6}\rceil\left(2\lfloor\frac{y}{4}\rfloor + 1\right) + 2\lceil\frac{y}{4}\rceil\left(2\lfloor\frac{x}{6}\rfloor + 1\right) & if\ y \equiv 3\ (mod\ 4) \end{cases}$$

Hence the result follows.

Example 2.6

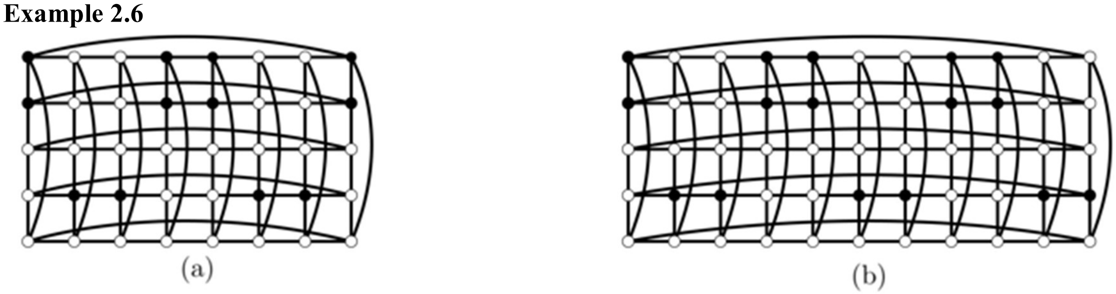


Figure 12: $(a)\ C_5 \times C_8$, $(b)\ C_5 \times C_{11}$

Here the darkened vertices denoted a CTCTD - set

In Figure 12 (a), $\gamma_{CTCTD}(C_5 \times C_8)$=$2\left(\frac{8}{4}\right)\left(2\left(\lceil\frac{5}{6}\rceil + \lfloor\frac{5}{6}\rfloor\right) + 1\right)$ =12.

In Figure 11 (b), γ_{CTCTD} $(C_5 \times C_{11})$=$2\lceil\frac{5}{6}\rceil\left(2\lfloor\frac{11}{4}\rfloor+1\right)+2\lceil\frac{11}{4}\rceil\left(2\lfloor\frac{5}{6}\rfloor+1\right)$=16

3 Conclusion

This article explored the CTCTD- number of graphs formed by the cartesian product of paths and cycles. We calculated exact values and established bounds, showing how the structure of these graphs affects domination and connectivity. Our findings help understand how to choose dominating vertices efficiently and can be applied to areas like network design and optimization. They also provide a foundation for future research on CTCTD-numbers in more complex graph types.

Acknowledgement

The research work was supported by UGC-SAP (DSA-I), Department of Mathematics, The Gandhigram Rural Institute - Deemed to be University, Gandhigram.

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