RESEARCH PAPER

# Comparative Evaluation of Classical Lagrangian Distributions for Pharmaceutical Count Data

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**Abstract**

Lagrangian distributions are a significant family of discrete distributions that have been extensively used to model count data from various fields such as branching process, reliability, epidemiology, actuarial studies, and pharmaceuticals. In this paper, a comparison between four popular Lagrangian distributions; the Borel, Haight, Consul, and Geeta distributions, is conducted in order to assess the applicability of each distribution to modelling count data from pharmaceuticals. The competing models' theoretical properties, which include probability mass function, parameter structure, and the nature of the hazard rate, are carefully discussed and compared. The competing models' hazard rates are analyzed in order to examine their reliability properties under different sets of parameters. The practical applicability of the distributions is tested using count data from pharmaceutical prescriptions, where model parameters are estimated using the maximum likelihood estimation technique. The goodness-of-fit measures include the negative log-likelihood, AIC, BIC, and Pearson Chi-Square statistics.The results from the comparison show that the Consul distribution model offers the best fit among all the models considered, as it gives the smallest values for all the goodness-of-fit statistics. The Borel and Geeta distribution models also give good fits, but the Haight distribution model shows relatively poor fit among the other models considered. From the hazard rate analysis, it is clear that the classical Lagrangian distributions are mainly characterized by either decreasing hazard rates or constant hazard rates. This makes the classical Lagrangian distributions appropriate for count processes whose conditional probability of occurrences decreases.

***Keywords:*** Lagrangian distributions; Pharmaceutical count data; Borel distribution; Consul distribution; Geeta distribution; Haight distribution; Maximum likelihood estimation; Goodness-of-fit; Hazard rate function.

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## *I*ntroduction

Count data occur naturally in several scientific fields such as medicine, epidemiology, biology, actuarial science, reliability engineering, ecology, and drug research among others. Such data constitute the number of events that have occurred within a certain observational unit and are normally analyzed using various discrete probability distributions. The Poisson distribution has always been the main statistical distribution used to analyze count data due to its ease of calculation. But the basic assumption of the distribution whereby the mean is equal to the variance is often violated due to the fact that most observed datasets display over-dispersion, under-dispersion, skewness, or clustering.Consequently, a variety of flexible discrete distributions have been developed to model complex count phenomena more accurately (Johnson et al., 2005; Nakagawa & Osaki, 1975; Shmueli et al., 2005; Dean & Nielsen, 2007; Cameron & Trivedi, 2013; Hilbe, 2014; Hassan et al., 2020,Kemp & Kemp, 1965).

The Lagrangian distribution is one of the well-known families of discrete distributions that can be obtained by means of the Lagrange expansion theorem. The mathematical background of such distributions dates back to the studies of Lagrange (1770), whereas the probabilistic derivation is due to Consul and Shenton (1972, 1975). Through proper selection of the generating function, the Lagrangian approach allows for obtaining various discrete probability distributions. Due to their mathematical simplicity and modeling capacity, such distributions have been applied in many different fields, such as branching processes, queuing models, reliability theory, biology, epidemiology, insurance and others where count data appear (Consul, 1989; Johnson et al., 2005,Steutel, F. & Hansen, Bjorn. (1989),Consul, P. C., & Famoye, F. (2006),Irshad et al., 2022,Irshad M et al., 2022,Irshad et al., 2023).

Some classical examples from this class include the Borel distribution proposed by Borel (1942),

which is one of the earliest Lagrangian distributions that have been extensively used for branching process problems and cluster count data. Another classical Lagrangian model has been proposed by Haight , who proposed this distribution for modeling stochastic count process problems as well as in queueing. Later, Consul and Shenton (1975) extended the family of Lagrangian distributions through some flexible parameterizations that enable modeling of wider classes of count data. The development and applications of generalized Lagrangian and generalized Poisson distributions have been reviewed by Consul (1989).

Count data is prevalent in pharmaceutical research when considering studies of prescription patterns, number of prescribed medicines per patient, adverse effects of medicines, medical prescription errors, hospitalizations and use of health care facilities. Correct statistical analysis of these variables is critical in assessing the behavior of prescribing practices, quality of prescription, efficient allocation of health care resources and evidence-based medicine decisions. As pharmaceutical count data violate the assumptions of Poisson distribution, count models that are flexible enough for the statistical analysis offer better choice (Cameron and Trivedi, 2013; Hilbe, 2014). Pharmaceutical prescription data set used in the current study and described in Idris et al. (2024) is a real-life example of such count data.

Though many Lagrangian distributions have been suggested, much research conducted so far has concentrated on presenting specific distributions and analyzing them mathematically. In recent years, several new flexible discrete distributions have also been developed to improve the modelling of count data exhibiting complex dispersion characteristics (Hassan et al., 2020,Sellers, K. F., & Raim, A. (2016)).However, the comparative studies that examine their theoretical features, hazard rate behavior, and performance using actual data on counts from pharmaceuticals is rather scarce. This is important since such analysis provides some recommendations on choosing suitable models and highlights advantages and disadvantages of the considered Lagrangian distributions.

Taking these into account, the present paper attempts to compare four classical Lagrangian distributions that include Borel, Haight, Consul, and Geeta distributions. The probability distribution, parameterization, and hazard function of these distributions have been studied in detail. Their performances have been assessed by means of maximum likelihood estimation in an application with the help of a pharmaceutical prescription count data set. Adequacy of these models has been checked by negative log-likelihood, Akaike Information Criterion, Bayesian Information Criterion, and Pearson's chi-square test statistics. These findings will give us valuable information about the suitability of these classical Lagrangian distributions for pharmaceutical data.

## 1. Some Foundational Concepts Related to Lagrangian Distributions

Lagrangian distributions form an important class of discrete probability distributions generated through the application of the Lagrange expansion theorem. These distributions provide a flexible framework for modelling count data and have found applications in diverse fields such as biology, epidemiology, insurance, reliability theory, and pharmaceutical sciences. The construction of Lagrangian distributions is based on a functional equation whose solution yields a probability generating function and, consequently, a probability distribution.

### *1.1. Lagrange Expansion Theorem*

The Lagrange expansion theorem plays a central role in the development of Lagrangian probability distributions. Let

$z = u\,g(z)$,

where $g(z)$ is an analytic function in a neighbourhood of the origin. If $z = \psi(u)$ denotes the smallest root of the above equation, then the Lagrange expansion theorem gives

$$\psi(u) = \sum_{x=1}^{\infty} \frac{u^x}{x!} \frac{d^{x-1}}{dz^{x-1}} [g(z)]^x \Big|_{z=0} . \quad (2.1)$$

This result provides a systematic approach for constructing probability generating functions and discrete probability distributions.

### *1.2. Basic Lagrangian Distribution*

Let $g(z)$ be a continuously differentiable function satisfying

$g(1) = 1$ and $g(0) \neq 0$.

Consider the transformation

$z=ug(z)$.

If the coefficients generated by the Lagrange expansion are non-negative, that is,

$$\frac{d^{x-1}}{dz^{x-1}} [g(z)]^x \Big|_{z=0} \geq 0, \qquad x = 1, 2, \ldots,$$

then the smallest root $z = \psi(u)$ defines a valid probability generating function. Consequently,

$$\psi(u) = \sum_{x=1}^{\infty} \frac{u^x}{x!} \frac{d^{x-1}}{dz^{x-1}} [g(z)]^x \Big|_{z=0}$$

(2.2)

is the probability generating function of the corresponding Basic Lagrangian Distribution.

The associated probability mass function is

$$P(X = x) = \frac{1}{x!}\frac{d^{x-1}}{dz^{x-1}}[g(z)]^x\Big|_{z=0}, \qquad x = 1, 2, \ldots,$$

(2.3)

The Basic Lagrangian Distribution serves as the fundamental building block for numerous generalized Lagrangian models developed in the literature. By choosing different forms of the generating function $g(z)$, a wide variety of count distributions can be obtained. These distributions are capable of modelling different dispersion structures and therefore provide greater flexibility than classical count models such as the Poisson distribution. Several well-known discrete probability distributions can be derived from the Lagrangian framework by selecting suitable generating functions. These distributions have been extensively studied in the literature and provide flexible models for count data exhibiting different dispersion characteristics. In this section, some important classical Lagrangian distributions relevant to count data modelling are briefly reviewed.

### 1.3. *Borel Distribution*

The Borel distribution was introduced by Borel (1942) and is obtained by choosing

$$g(z) = e^{\lambda(z-)}, \qquad 0 < \lambda < 1.$$

The probability mass function is

$$P(X = x) = \frac{(\lambda x)^{x-1}}{x!}e^{-\lambda x}, \qquad x = 1, 2, \ldots$$

The mean and variance are

$$E(X) = \frac{1}{1-\lambda},$$

$$Var\,(X) = \frac{\lambda}{(1-\lambda)^3}.$$

The Borel distribution is commonly used to model branching processes and clustered count phenomena.

### 1.4. *Haight Distribution*

Haight (1961) proposed a Lagrangian distribution generated by

$g(z) = q(1 - pz)^{-1}$,

where

$0 < p = 1 - q < 1$.

The probability mass function is

$$P(X = x) = \frac{1}{2x-1}\binom{2x-1}{x}q^x p^{x-1}$$

This distribution has applications in queueing theory and stochastic counting processes.

### 1.5. *Consul Distribution*

The Consul distribution was introduced by Consul and Shenton (1975). The generating function is

$g(z) = (1 - \theta + \theta z)^m$,

where

$0 < \theta < 1$, $\quad m \in N$.

Its probability mass function is

$$P(X = x) = \frac{1}{x}\binom{mx}{x-1}\theta^{x-1}(1-\theta)^{mx-x+1}, \quad x = 1, 2, \ldots$$

The distribution provides greater flexibility than classical Poisson models and is useful for modelling over-dispersed count data.

### 1.6. *Geeta Distribution*

The Geeta distribution was proposed by Consul (1990) with generating function

$$g(z) = (1-\theta)^{m-1}(1-\theta z)^{1-m},$$

Where

$0 < \theta < 1$, $\quad 1 < m < \theta^{-1}$.

The corresponding probability mass function is

$$P(X = x) = \frac{1}{mx-1}\binom{mx-1}{x}\theta^{x-1}(1-\theta)^{mx-x}, \qquad x = 1, 2, \ldots$$

The Geeta distribution extends the family of Lagrangian count distributions and can accommodate a variety of count data structures.

Table 1: Summary of Classical Lagrangian Distributions

| Distribution | Parameters | Support | Main Application |
|---|---|---|---|
| Borel | $\lambda$ | $x = 1, 2, \ldots$ | Branching processes |
| Haight | $p$, $q$ | $x = 1, 2, \ldots$ | Queueing systems |
| Consul | $\theta, m$ | $x = 1, 2, \ldots$ | Over-dispersed counts |
| Geeta | $\theta, m$ | $x = 1, 2, \ldots$ | Flexible count modelling |

## 3. Comparative Characteristics of Classical Lagrangian Distributions

The classical Lagrangian distributions discussed in the previous section represent some of the earliest and most important members of the Lagrangian family of discrete probability distributions. Although all these distributions originate from the Lagrange expansion framework, they differ considerably in terms of their generating functions, parameter structures, probability mass functions, and modelling

capabilities. A comparative study of these distributions provides useful insights into their suitability for analysing count data arising in practical applications.

The Borel distribution is one of the simplest Lagrangian distributions and is characterized by a single parameter. It is primarily used in branching pro-cess theory and cluster-type count data. The Haight distribution extends the Lagrangian framework through a different generating function structure and has applications in stochastic processes and queueing systems. The Consul distribution introduces additional flexibility through two parameters, enabling the modelling of a wider range of count data patterns. Similarly, the Geeta distribution provides an alternative parameterization that allows greater adaptability in modelling count data exhibiting varying dispersion characteristics.

A summary of the major characteristics of these distributions is presented in Table 2.

Table 2:Comparative Characteristics of Classical Lagrangian Distributions

| Distribution | Parameters | Support | Complexity | Flexibility |
|---|---|---|---|---|
| Borel | $\lambda$ | $x = 1, 2, \ldots$ | Low | Moderate |
| Haight | $p, q$ | $x = 1, 2, \ldots$ | Moderate | Moderate |
| Consul | $\theta, m$ | $x = 1, 2, \ldots$ | Moderate | High |
| Geeta | $\theta, m$ | $x = 1, 2, \ldots$ | Moderate | High |

Table 2 shows that the Borel distribution possesses the simplest structure among the four distributions considered. However, its single-parameter formulation restricts its flexibility when modelling highly dispersed count data. The Haight distribution offers additional structural complexity but remains relatively limited in practical applications. In contrast, the Consul and Geeta distributions provide greater flexibility through their richer parameterization and are capable of modelling a wider variety of count data behaviours.

These classical Lagrangian distributions form the theoretical foundation for numerous modern extensions that have been proposed in the literature. Their study is particularly relevant in pharmaceutical count data analysis, where observations frequently exhibit skewness, over-dispersion, or other departures from the assumptions of the classical Poisson distribution. Understanding the characteristics of these distributions facilitates the selection of appropriate models and provides a basis for comparative evaluation using real-world pharmaceutical datasets.

## 4. Hazard Rate Function

The hazard rate function (HRF) is an important reliability characteristic that describes the conditional probability pf occurrence of an event at count x, given that it has not occurred before x. For a discrete random variable X with probability mass function $p(x)$ and survival function $S(x)=P(X \geq x)$, the hazard rate function is defined as

$$h(x) = \frac{p(x)}{S(x)}, \qquad x = 1, 2, \ldots$$

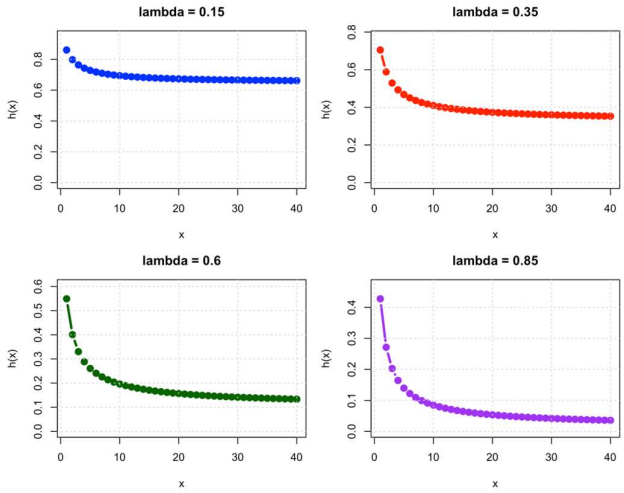


Figure 4.1: Representative HRF Shapes of the Borel Distribution

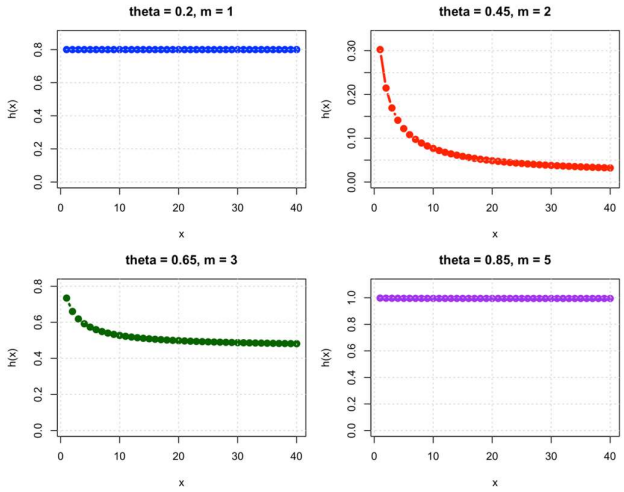


Figure 4.2: Representative HRF Shapes of the Consul Distribution

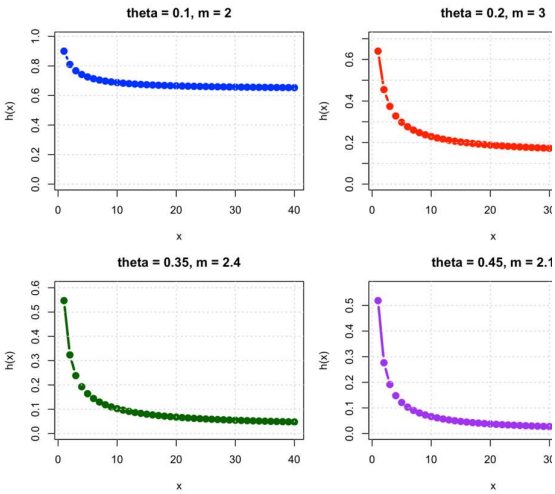


Figure 4.3: Representative HRF Shapes of the Geeta Distribution

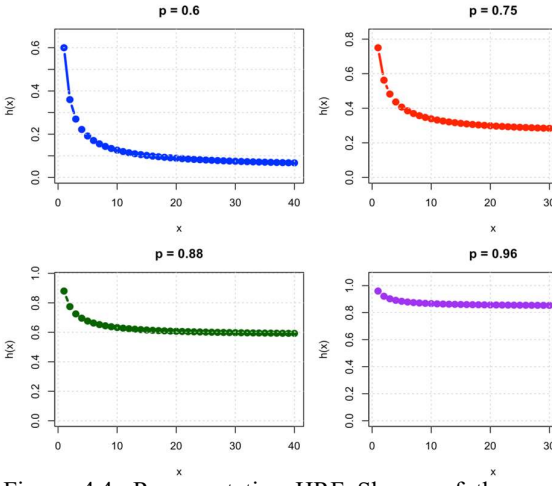


Figure 4.4: Representative HRF Shapes of the Haight Distribution

The HRF is really useful for understanding how things age and what makes them fail. So looking at the hazard function is very important when we want to know if a probability model is good enough for things, like branching processes, biological populations, epidemiology, reliability engineering and actuarial science.

For the Borel distribution the hazard function goes down smoothly for all the values of the parameter λ we consider. When λ is small the hazard is higher all the time.. When λ gets bigger the whole hazard curve goes down and starts to decrease faster. The hazard function eventually stops at a positive value, which means the risk gets smaller as the count gets bigger.

The Consul distribution demonstrates comparatively greater flexibility. Depending on the parameter combination (θ,m)the hazard function exhibits either an approximately constant hazard or a decreasing hazard. For example, when (θ,m)=(0.2,1) and (0.85,5), the hazard remains nearly constant throughout the support, whereas other parameter combinations produce a monotone decreasing hazard. This behaviour indicates that the Consul distribution can accommodate both constant-risk and decreasing-risk count processes.

The Geeta distribution consistently exhibits decreasing hazard rate behaviour for all selected parameter values. Increasing the parameter θ results in a steeper decline of the hazard function and reduces the limiting hazard level. The parameter mm influences the rate of decline, thereby providing additional flexibility in modelling count data with varying degrees of hazard reduction.

Similarly, the Haight distribution produces decreasing hazard functions for all examined values of the parameter pp. However, as pp approaches unity, the hazard function becomes progressively flatter and eventually behaves almost like a constant hazard function. Thus, higher values of p correspond to more persistent hazard levels across the support.

Overall, the hazard rate analysis indicates that the existing classical Lagrangian distributions are primarily suitable for modelling count data characterized by decreasing or nearly constant hazard behaviour. None of the considered distributions exhibit increasing, bathtub-shaped, or upside-down bathtub hazard functions for the selected parameter settings. Consequently, although these models adequately describe several practical count processes with diminishing event probabilities, their flexibility in representing more complex hazard structures remains limited. This limitation provides further motivation for the development of more flexible Lagrangian count distributions capable of accommodating a wider variety of hazard rate behaviours.

## 5. Data Analysis

### *5.1 Description of Dataset*

This research studies the pharmaceutical count data extracted from writings by Idris et al. (2024), which reviewed drug prescribing patterns and prescription details using evidence from 615 healthcare facility prescriptions. The present study uses the drugs per prescription as the response variable. Each case corresponds to the total number of drugs in one prescription. The number of drugs per prescription was between 1 and 8, totaling 1573 drugs for all prescriptions. As the response variable consists of counts of positive integers, this dataset can be modeled as a count data. The frequency distribution of the number of

drugs prescribed per prescription is shown is Table 3.

Table 3. Frequency Distribution of the Number of Drugs Prescribed per Prescription

| **Number of Drugs ((x))** | **Observed Frequency ((f))** |
|---|---|
| 1 | 176 |
| 2 | 159 |
| 3 | 124 |
| 4 | 100 |
| 5 | 38 |
| 6 | 12 |
| 7 | 3 |
| 8 | 3 |
| **Total** | **615** |

*5.2 Descriptive Statistical Analysis*

The descriptive statistics of the pharmaceutical count data are presented in Table 4. These measures provide an overview of the distributional characteristics of the number of drugs prescribed per prescription before fitting the Lagrangian distributions.

Table 4. Descriptive Statistics of the Pharmaceutical Count Data

| **Statistic** | **Value** |
|---|---|
| Sample size ((n)) | 615 |
| Mean | 2.5577 |
| Variance | 1.9930 |
| Standard deviation | 1.4117 |
| Minimum | 1 |
| Maximum | 8 |
| Coefficient of Variation (%) | 55.20 |
| Skewness | 0.8134 |
| Kurtosis | 3.4083 |
| Index of Dispersion | 0.7792 |

The pharmaceutical count dataset contains 615 records, each corresponding to a prescription which contains between 1 and 8 drugs. The drugs per prescription had a mean of 2.56 and variance of 1.41, which is a measure of moderate variability. The coefficient of variation was equal to 55.20%, suggesting a moderate relative dispersion for the distribution. The positive skewness for the distribution was equal to 0.8134, indicating a longer right tail for the distribution, suggesting a small number of prescriptions contained a larger number of drugs. The kurtosis value for the distribution was equal to 3.4083, indicating a distribution that is slightly more peaked than a normal distribution. The Index of Dispersion was equal to 0.7792, suggesting an under-dispersion, which is a state where variance is smaller than the mean. The under-dispersion and the characteristics of the distribution that is being modeled supports a comparison of flexible Lagrangian count distributions, since the classical Poisson model will fail to capture the majority of the distribution's variability and characteristics.

**6. Parameter Estimation**

The parameters of the competing Lagrangian distributions are estimated by the method of maximum likelihood. Let x1, x2, . . . , xn denote the observed sample and let p(x; Θ) be the probability mass function of a distribution with parameter vector Θ. The likelihood function is given by

$L(\Theta)=\prod_{i=1}^{n} \quad p(x_i;\, \theta)$,

and the corresponding log-likelihood function is

$$l(\,\theta) = \sum_{i-1}^{n} \quad log\ log\ (px_i;\,)\,.$$

The maximum likelihood estimates are obtained by solving the likelihood equations or, when closed-form solutions are not available, by numerical maximization of the log-likelihood function.

*6.1. Borel Distribution*

The probability mass function of the Borel distribution is

$$P(X = x) = \frac{(\lambda x)^{x-1}}{x!} e^{-\lambda x}, \quad x = 1, 2, \ldots, 0 < \lambda < 1.$$

The likelihood function is

$$L(\lambda) = \prod_{i=1}^{n} \quad \frac{(\lambda x)^{x-1}}{x!} e^{-\lambda x}$$

Differentiating with respect to $\lambda$ and equating the score function to zero gives

$$\hat{\lambda} = \frac{\sum_{i=1}^{n} x_i - n}{\sum_{i=1}^{n} x_i}$$

*6.2.Haight Distribution*

The probability mass function of the Haight distribution is

$P(X = x) = \frac{1}{2x-1}\left(\frac{2x-1}{x}\right) q^x p^{x-1}$, $x = 1, 2, \ldots$

where

$0<p=1-q<1$.

Hence, the probability mass function can be written as

$P(X = x) = \frac{1}{2x-1}\left(\frac{2x-1}{x}\right)(1-p)^x p^{x-1}$.

The likelihood function is

$L(p) = \prod_{i=1}^{n} \left[\frac{1}{2x_i-1}\left(\frac{2x_i-1}{x_i}\right)(1-p)^{x_i} p^{x_i-1}\right]$.

The corresponding log-likelihood function is

$\ell(p) = \sum_{i=1}^{n} \left[-log\ log\ (2x_i - 1) + log\left(\frac{2x_i-1}{x_i}\right) + x_i\ log\ log\ (1-p) + (x_i - 1) log p\right]$.

Differentiating with respect to $p$ and equating the score function to zero gives

$\hat{p} = \frac{\sum_{i=1}^{n} (x_i-1)}{2\sum_{i=1}^{n} x_i - n} = \frac{\sum_{i=1}^{n} x_i - n}{2\sum_{i=1}^{n} x_i - n}$.

Since $q = 1 - p$, the estimate of $q$ is

$\hat{q} = 1 - \hat{p}$.

*6.3. Consul Distribution*

The probability mass function of the Consul distribution is

$$P(X = x) = \frac{1}{x}\left(\frac{mx}{x-1}\right)\theta^{x-1}(1-\theta)^{mx-x+1}, \quad x = 1, 2, \ldots$$

where

$0 < \theta < 1$, $m \in N$.

The likelihood function is

$L(\theta, m) = \prod_{i=1}^{n} \left[\frac{1}{x_i}\left(\frac{mx_i}{x_i-1}\right)\theta^{x_i-1}(1-\theta)^{mx_i-x_i+1}\right]$.

The log-likelihood function is

$\ell(\theta, m) = \sum_{i=1}^{n} \left[-log\ log\ x_i + log\left(\frac{mx_i}{x_i-1}\right) + (x_i - 1)\ log\ log\ \theta + (mx_i - x_i + 1)\ log\ log\ (1-\theta)\right]$.

For fixed m, differentiating with respect to θ gives and equating the score function to zero gives

$\hat{\theta} = \frac{\sum_{i=1}^{n} (x_i-1)}{m\sum_{i=1}^{n} x_i} = \frac{\sum_{i=1}^{n} x_i - n}{m\sum_{i=1}^{n} x_i}$.

The derivative of the log-likelihood with respect to $m$ is

$\frac{\partial l(\theta,m)}{\partial m} = \sum_{i=1}^{n} x_i[\psi(mx_i + 1) - \psi(mx_i - x_i + 2) + log(1-\theta)]$,

where $\psi(\cdot)$ denotes the digamma function. Since this equation does not generally yield a closed-form solution for $m$, the MLE of $m$ is obtained numerically. When m is treated as an integer-valued parameter, the log-likelihood may be evaluated over a suitable set of positive integer values of $m$, and the value giving the maximum log-likelihood is selected.

*6.4. Geeta Distribution*

The probability mass function of the Geeta distribution is

$$P(X = x) = \frac{1}{mx-1}\left(\frac{mx-1}{x}\right)\theta^{x-1}(1-\theta)^{mx-x}, \quad x = 1, 2, \ldots,$$

where

$0 < \theta < 1$, $1 < m < \theta^{-1}$.

The likelihood function is

$L(\theta, m) = \prod_{i=1}^{n} \left[\frac{1}{mx_i-1}\left(\frac{mx_i-1}{x_i}\right)\theta^{x_i-1}(1-\theta)^{mx_i-x_i}\right]$.

The log-likelihood function is

$\ell(\theta,m) = \sum_{i=1}^{n} \left[-log\ log\ (mx_i - 1) + log\left(\frac{mx_i-1}{x_i}\right) + (x_i - 1)\ log\ log\ \theta + (mx_i - x_i)\ log\ log\ (1-\theta)\right]$.

Using the gamma function representation,

$$\left(\frac{mx_i - 1}{x_i}\right) = \frac{\Gamma(mx_i)}{\Gamma(x_i+1)\Gamma(mx_i - x_i)}$$

The log-likelihood becomes

$\ell(\theta,m) = \sum_{i=1}^{n} [-log\ log\ (mx_i - 1) + log\ \Gamma(mx_i) - log\Gamma(x_i + 1) - log\Gamma(mx_i - x_i)(1-\theta)]$

For fixed m, differentiating with respect to θ and equating the score function to zero gives

$$\hat{\theta} = \frac{\sum_{i=1}^{n} (x_i - 1)}{m\sum_{i=1}^{n} x_i - n} = \frac{\sum_{i=1}^{n} x_i - n}{m\sum_{i=1}^{n} x_i - n}$$

The derivative of the log-likelihood with respect to m is

$$\frac{\partial l(\theta,m)}{\partial m} = \sum_{i=1}^{n} [-\frac{x_i}{mx_i - 1} + x_i\psi(mx_i) - x_i\psi(mx_i - x_i) + x_i log(1-\theta)]$$

Since this score equation is nonlinear and does not have a simple closed-form solution, the MLE of m is obtained numerically subject to the parameter restriction

$1 < m < \theta^{-1}$.

## 7. Goodness-of-Fit Measures

In this section, Lagrangian distributions are evaluated based on some popular goodness-of-fit measures. The maximum likelihood estimates of a model's parameters are evaluated using negative log-likelihood, the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Pearson's chi-square statistic. These

measures assess the adequacy of a model's fit, considering the balance between the quality of fit, and the model's complexity.

The best model describing the pharmaceutical count data from Table 3 is the distribution of Lagrangian count data with the smallest negative log-likelihood, AIC, and BIC, and the best Pearson's chi-square statistic.

Table 5. Final Goodness-of-Fit Comparison

| Distribution | Parameter Estimates | Loh-Likelihood | AIC | BIC | Pearson $\chi^2$ |
|---|---|---|---|---|---|
| Borel | $\lambda$=0.6090 | -1172.134 | 2346.269 | 2350.690 | 345.540 |
| Haight | P=0.3785, q=0.6215 | -1254.021 | 2510.042 | 2514.463 | 583.013 |
| Consul | $\theta$=0.6090, m=1 | -1054.637 | 2109.274 | 2118.118 | 84.097 |
| Geeta | $\theta$=0.0126, m=48.7328 | -1174.129 | 2352.258 | 2361.102 | 350.088 |

Table 5 summarizes goodness-of-fit statistics for the four competing Lagrangian distributions fitted to the pharmaceutical prescription count data. The performance of the models was assessed using the negative log-likelihood (-log L), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Pearson's chi-square ( $\chi^2$) statistic. The Consul distribution fitted the best overall performance among the fitted models, generating the smallest values of (-log L) (1052.637), AIC (2109.274), BIC (2118.118) and Pearson's chi-square statistic (84.097). The Borel and Geeta distributions performed moderately with similar goodness-of-fit values, while the Haight distribution had the largest values for all criteria, indicating the poorest fit. Based on these model selection criteria, the Consul distribution is the best fit model for the pharmaceutical prescription count data analyzed in this study.

## 8. Conclusion

This study provides a comparison among four classical Lagrangian distributions such as the Borel, Haight, Consul, and Geeta distributions, which were used to model pharmaceutical count data. The properties of these distributions, their hazard rate functions, and empirical performance were analyzed. The analysis of hazard rate functions shows that these distributions mostly have decreasing or almost constant hazard rates, which means that they can be applied to modeling count data with decreasing conditional probabilities.

The empirical performance of the competing distributions was evaluated using the pharmaceutical prescription count data by employing the method of maximum likelihood estimation and goodness-of-fit criteria. The Consul distribution demonstrated the best performance in the case of the application of all criteria since it yielded the lowest values of the negative log-likelihood, AIC, BIC, and Pearson's chi-square statistics. The Borel and Geeta distributions exhibited reasonable fit, while the Haight distribution had poor performance compared to the others.

In conclusion, the evidence obtained in this study shows that Lagrangian distributions are valuable options for modelling count data in the pharmaceutical sector. Comparisons made in this study can help researchers choose appropriate Lagrangian distributions and serve as a standard against which future methods will be developed. In further research, attention can be paid to extending the family of Lagrangian distributions to handle more complicated hazard rates and dispersion characteristics of count data.

## 9. References


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