

A Two-Prey One-Predator System with Saturated Fear Effect and Intraspecific Competition Stability and Saddle-Node Bifurcation

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ABSTRACT

In this article, a prey-predator model consisting of two prey species and one predator species is proposed. Intraspecific competition is considered within each population, including both prey and predator species, while the interactions between the prey and predator are described using two different functional responses. The positivity and boundedness of the solutions are established, ensuring that the population densities remain biologically meaningful and finite. The local stability of the equilibrium points is analysed and the system is shown to be locally stable in the vicinity of the equilibrium points under suitable conditions. Furthermore, saddle-node bifurcation occurs at two equilibrium points, indicating a qualitative change in the population dynamics as the system parameters vary.

Keywords: Beddington, functional, Lotka-Volterra, saturated fear effect, bifurcation.

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1 INTRODUCTION

Prey-predator interactions are important components of ecological systems and have been extensively studied using mathematical models. Such models help describe the interactions between prey and predator populations and provide insight into their persistence, coexistence, stability and dynamical behaviour. The complexity of prey-predator systems can be increased by incorporating biologically relevant factors such as nonlinear functional responses, fear effects, intraspecific competition and multiple prey or predator populations.

Cai [2] studied a stage-structured prey-predator model with a Beddington-DeAngelis functional response. The Beddington-DeAngelis functional response accounts for predator interference in addition to prey density and provides a realistic description of prey-predator interactions. Pal [8] investigated the role of fear in a prey-predator model with a Beddington-DeAngelis functional response. Panja [11] studied a prey-predator model with Beddington DeAngelis functional response, omnivore and predator switching. Shao and Kong [14] considered a prey-predator model with Beddington-DeAngelis functional response and multiple delays in deterministic and stochastic environments. Wang and Gao [15] further investigated a prey-predator model with a Beddington-DeAngelis functional response. These studies indicate that the choice of functional response can significantly influence prey-predator dynamics.

Holling type IV functional responses have also been considered to describe nonlinear prey-predator

interactions. Yu et al. [17] studied a prey-predator model with a Holling type IV functional response. Kalra and Kaur [7] investigated a prey-predator model with a Beddington-DeAngelis Holling type IV functional response. More recently, Ali [1] studied a prey-predator model with a Holling type IV functional response. These studies demonstrate the importance of nonlinear functional responses in determining the dynamical behaviour of prey-predator systems.

The fear effect has received considerable attention in prey-predator modelling because the presence of predators can alter prey behaviour and reproduction even without direct predation. Wang [16] introduced the fear effect into prey predator interactions, while Pal et al. [8] incorporated fear into a prey-predator model with a Beddington-DeAngelis functional response. Gao and Dai [4] studied a prey-predator model incorporating delay and fear effects. Sha and Chattopadhyay [13] investigated the fear effect in a prey-predator model with disease in the predator population. Ishaque [6] studied a prey-predator model based on the fear effect with bifurcation analysis and chaos control.

The effect of fear may not increase indefinitely with increasing predator density and saturated fear effect can provide a more realistic representation of predator induced behavioural responses. Zhu and Zhou [18] studied the stability of a prey-predator model with a saturated fear cost effect. He et al. [5] investigated the influence of saturated fear effects on species dynamics using a Lotka-Volterra competition model. These studies

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show that both fear and saturated fear effects can substantially influence population dynamics.

Intraspecific competition is another important factor influencing population dynamics. Individuals of the same species compete for limited resources such as food, space and reproductive opportunities. Pal [9] investigated cooperation and intraspecific competition of prey in a prey-predator model with refuge. He et al. [5] also considered species competition together with saturated fear effects in one prey and one predator. These studies highlight the importance of incorporating intraspecific competition when analysing ecological systems. Prey-predator models involving multiple populations have also been investigated. Das et al. [3] analysed a two-predator-one-prey model with predator advantage.

Bifurcation analysis plays an important role in identifying qualitative changes in prey-predator dynamics. Sen [12] studied bifurcations in a prey predator model with the Allee effect in the predator population. Panigoro [10] investigated bifurcation behaviour in a prey-predator model involving age structure and intraspecific competition. These studies highlight the importance of bifurcation analysis in understanding changes in prey-predator dynamics.

Although several studies have investigated functional responses, fear effects, saturated fear effects, intraspecific competition, multiple prey or predator populations and bifurcation behaviour, these factors have generally been considered separately or in different combinations. In particular, there is scope for studying a two-prey-one-predator system in which intraspecific competition occurs in all three populations, the two prey-predator interactions are represented by two different functional responses and the prey populations are subjected to a saturated fear effect.

2 MODEL FORMULATION

Intraspecific competition represents competition among individuals of the same species for limited resources and is considered in both prey and predator populations. The first prey-predator interaction is described by a Holling type IV functional response associated with a saturated fear effect, which represents the reduction in prey growth or reproduction due to the perceived risk of predation. The second prey-predator interaction is described by a Beddington-DeAngelis functional response, which accounts for predator interference in addition to prey density. These mechanisms together provide a realistic representation of the interactions among the two prey populations and the predator. Based on the above assumptions, the mathematical model is formulated as follows:

$$\frac{dG_1}{dt} = C_1 G_1 \left(\Gamma + \frac{1 - \Gamma}{(1 + \omega G_3)} - \frac{G_1}{e_1} \right) - \frac{\delta G_1 G_3}{(G_1^2 + S^2)} - \mathfrak{P} G_1^2,$$

$$\frac{dG_2}{dt} = C_2 G_2 \left(1 - \frac{G_2}{e_2} \right) - \frac{d G_2 G_3}{(\gamma + G_2 + Q G_3)} - n G_2^2, \quad (2.1)$$

$$\frac{dG_3}{dt} = \frac{U_1 \delta G_1 G_3}{(G_1^2 + S^2)} + \frac{U_2 d G_2 G_3}{(\gamma + G_2 + Q G_3)} - m G_3 - h G_3^2.$$

Table 1: Biological Meaning of the Parameters used in the Model

Parameters	Description
G_1	Prey population with saturated fear effect
G_2	Prey population with Beddington-DeAngelis functional response
G_3	Predator population
Γ	Normalization coefficient
Q	Saturation constant associated with Beddington- DeAngelis functional response
ω	Fear level
$\bar{\Gamma}$	Saturation value of the fear effect
$\bar{S}$	Half saturation constant
M	Death rate of predator
e_1, e_2	Carrying capacity
$\hat{\rho}$	Interaction coefficient of G_1
$\hat{d}$	Interaction coefficient of G_2
U_1, U_2	Conversion efficiency of consumed preys into predators population growth
P	Intraspecific competition rate with in the prey population G_1
N	Intraspecific competition rate with in the prey population G_2
h	Intraspecific competition rate with in the predator population

3 Analysis of Positivity and Boundedness Properties of the System

Theorem 3.1. If it is non-negative for all 't', then all solutions of the system (2.1).

Proof:

By integrating each equation of the system (2.1) provide the following results,

$$G_1(t) = G_1(0) \exp \left[\int \left(C_1 \left(\Gamma + \frac{1 - \Gamma}{(1 + \omega G_3)} - \frac{G_1}{e_1} \right) - \frac{\delta G_3}{(G_1^2 + S^2)} - \mathfrak{P} G_1 \right) dt \right],$$

$$G_2(t) = G_2(0) \exp \left[\int \left(C_2 \left(1 - \frac{G_2}{e_2} \right) - \frac{d G_3}{(\gamma + G_2 + Q G_3)} - n G_2 \right) dt \right],$$

$$G_3(t) = G_3(0) \exp \left[\int \left(\frac{U_1 \delta G_1}{(G_1^2 + S^2)} + \frac{U_2 d G_2}{(\gamma + G_2 + Q G_3)} - m - h G_3 \right) dt \right].$$

$$(G_1^2 + S^2) - (\gamma + G_2 + QG_3)$$

Since, $G_1(0) \geq 0$, $G_2(0) \geq 0$, $G_3(0) \geq 0$ as $t \geq 0$.

Therefore, the system provided solutions that are non-negative for all $t \geq 0$.

Lemma 3.2. Assume that ϕ be an absolutely continuous function that satisfies the differential inequality

$$\frac{d\phi(t)}{dt} + \alpha_1 \phi(t) \leq \alpha_2 \quad \forall t \geq 0, \text{ where } (\alpha_1, \alpha_2) \in \mathbb{R}_+^3$$

$\alpha \neq 0$, then $\forall t \geq T \geq 0$.

$$\phi(t) \leq \frac{\alpha_1}{\alpha_2} - \left(\frac{\alpha_2}{\alpha_1} \phi(t) \right) e^{-\alpha_1(t-T)}.$$

Theorem 3.3. If $\Gamma > 1$, then the trajectories of system (2.1) are bounded.

Proof:

Define the function $\tilde{\mu} = G_1 + G_2 + G_3$ and differentiate $\tilde{\mu}$ with respect to 't', we obtain

$$\begin{aligned} \frac{d\tilde{\mu}}{dt} &= \frac{dG_1}{dt} + \frac{dG_2}{dt} + \frac{dG_3}{dt}, \\ &= C_1 G_1 \left(\Gamma + \frac{1 - \Gamma}{(1 + \omega G_3)} \right) + C_2 G_2 - \frac{C_1 G_1^2}{e_1} - \frac{C_2 G_2^2}{e_2} - m G_3 \\ &\quad - (1 - U_1) \frac{\partial G_1 G_3}{(G_1^2 + S^2)} - (1 - U_2) \frac{dG_2 G_3}{(\gamma + G_2 + QG_3)} \end{aligned}$$

$$- \mathfrak{P} G_1^2 - n G_2^2 - h G_3^2.$$

Since the conversion efficiencies satisfy $0 < U_1, U_2 \leq 1$, the two predation terms are non-positive. Also, since $\Gamma > 1$ and $1 + \omega G_3 \geq 1$, we have $(1 - \Gamma)/(1 + \omega G_3) \leq 0$, and therefore $\Gamma + (1 - \Gamma)/(1 + \omega G_3) \leq \Gamma$. Hence,

$$\frac{d\tilde{\mu}}{dt} \leq C_1 \Gamma G_1 - \frac{C_1 G_1^2}{e_1} + C_2 G_2 - \frac{C_2 G_2^2}{e_2} - m G_3 - \mathfrak{P} G_1^2 - n G_2^2 - h G_3^2.$$

For an arbitrary positive real number β with $0 < \beta \leq m$, adding $\beta \tilde{\mu} = \beta(G_1 + G_2 + G_3)$ to both sides gives

$$\begin{aligned} \frac{d\tilde{\mu}(t)}{dt} + \beta \tilde{\mu}(t) &\leq (C_1 \Gamma + \beta) G_1 - \frac{C_1 G_1^2}{e_1} + (C_2 + \beta) G_2 - \frac{C_2 G_2^2}{e_2} \\ &\quad - (m - \beta) G_3 - \mathfrak{P} G_1^2 - n G_2^2 - h G_3^2. \end{aligned}$$

$$\frac{d\tilde{\mu}(t)}{dt} + \beta \tilde{\mu}(t) \leq \frac{e_1}{4C_1} (C_1 \Gamma + \beta)^2 + \frac{e_2}{4C_2} (C_2 + \beta)^2.$$

Here we used $0 < \beta \leq m$, so that the terms $-(m - \beta)G_3$, $-\mathfrak{P}G_1^2$, $-nG_2^2$ and $-hG_3^2$ are non-positive. Moreover, the quadratic $(C_1 \Gamma + \beta)G_1 - C_1 G_1^2/e_1$ attains its maximum value $e_1(C_1 \Gamma + \beta)^2/(4C_1)$ at $G_1 = e_1(C_1 \Gamma + \beta)/(2C_1)$, and the quadratic $(C_2 + \beta)G_2 - C_2 G_2^2/e_2$ attains its maximum value $e_2(C_2 + \beta)^2/(4C_2)$ at $G_2 = e_2(C_2 + \beta)/(2C_2)$.

We define a constant $\eta \geq 0$ such that $\eta = \frac{e_1}{4C_1}(C_1\Gamma + \beta)^2 + \frac{e_2}{4C_2}(C_2 + \beta)^2$.

$$\frac{d\tilde{\mu}(t)}{dt} + \beta\tilde{\mu} \leq \eta.$$

Based on the above lemma 3.2 and letting $t \rightarrow \infty$, we get

$$\limsup_{t \rightarrow \infty} \tilde{\mu}(t) \leq \frac{\eta}{\beta}.$$

Thus, the system (2.1) of equation is bounded for the given initial condition.

4 Theoretical Analysis of the System

- (1) The trivial equilibrium point is $E_0(0, 0, 0)$.
- (2) The prey only existence equilibrium point is

$$E_1\left(\frac{C_1\Gamma e_1}{(C_1 + e_1\beta)}, \frac{C_2e_2}{(C_2 + e_2n)}, 0\right).$$

- (3) The second prey and predator existence equilibrium point $E_2(0, A^*, B^*)$ exists if $U_1\bar{d} > m + hB^*$, $e_2 > Q$, $\gamma > 1$ and $C_2Q(e_2 - Q) > de_2 + ne_2A^*$.

$$A^* = \frac{\gamma(m + h) + QB_*(m + hB^*)}{(U_2\bar{d} - m - hB^*)} \quad \text{and}$$

$$B^* = \frac{A^*e_2C_2(\gamma - 1) + A^{*2}(C_2 + ne_2) + ne_2\gamma}{(C_2Q(e_2 - Q) - de_2 - ne_2A^*)}.$$

- (4) The coexistence equilibrium point

$$E_3(G_1^*, G_2^*, G_3^*) \text{ exists if } C_2Q(e_2 - Q) > \bar{d}e_2 + ne_2G_2^*, \gamma > 1, e_2 > Q, U_2\bar{d} > m + hG_3^* \text{ and } (G_1^{*2} + \bar{S}^2)(\gamma + QG_3^*)(m + hG_3^*).$$

Where,

$$G_3^* = \frac{G_2^*e_2C_2(\gamma - 1) + G_2^{*2}(C_2 + ne_2) + ne_2\gamma}{(C_2Q(e_2 - Q) - \bar{d}e_2 - ne_2G_2^*)}$$

$$G_2^* = \frac{(G_1^{*2} + \bar{S}^2)[(\gamma + QG_3^*)(m + hG_3^*)] - U_1\hat{\phi}G_1^*(\gamma + QG_3^*)}{[U_1\hat{\phi}G_1^* + (G_1^{*2} + \bar{S}^2)(U_2\bar{d} - m - hG_3^*)]}$$

and $G_1^* = B_3 G_1^3 + B_2 G_1^2 + B_1 G_1 + B_0 = 0$,

where B_0, B_1, B_2, B_3 are constants depending on the parameters

$U_1, \Gamma, S, \beta, \omega, e_1$ and C_1 .

Hence, the G_1^* is positive root of the above polynomial.

5 Local Stability Analysis of the System

The system (2.1) variational matrix is,

$$J(G_1, G_2, G_3) = \begin{bmatrix} V_{11} & 0 & V_{13} \\ 0 & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix},$$

where

$$V_{11} = C_1 \Gamma + \frac{C_1(1-\Gamma)}{(1+\omega G_3)} - \frac{2C_1 G_1}{e_1} - \frac{\partial G_3}{(G_1^2 + S^2)} + \frac{2\partial G_1^2 G_3}{(G_1^2 + S^2)^2} - 2\beta G_1,$$

$$V_{12} = 0,$$

$$V_{13} = -\frac{C_1 \omega (1-\Gamma) G_1}{(1+\omega G_3)^2} - \frac{\partial G_1}{(G_1^2 + S^2)},$$

$$V_{21} = 0,$$

$$V_{22} = C_2 - \frac{2C_2 G_2}{e_2} - \frac{dG_3}{(\gamma + G_2 + QG_3)} + \frac{dG_2}{G_3} (\gamma + G_2 + QG_3)^2 - 2nG_2,$$

$$V_{23} = -\frac{dG_2}{(\gamma + G_2 + QG_3)} + \frac{QdG_2}{G_3} (\gamma + G_2 + QG_3)^2,$$

$$V_{31} = \frac{U_1 \partial G_3}{(G_1^2 + S^2)} + \frac{2U_1 \partial G_1^2 G_3}{(G_1^2 + S^2)^2},$$

$$V_{32} = \frac{U_2 dG_3}{(\gamma + G_2 + QG_3)} - \frac{U_2 dG_2}{G_3} (\gamma + G_2 + QG_3)^2,$$

$$V_{33} = \frac{U_1 \partial G_1}{(G_1^2 + S^2)} + \frac{U_2 dG_2}{(\gamma + G_2 + QG_3)} - \frac{QU_2 dG_2 G_3}{(\gamma + G_2 + QG_3)^2} - m - 2hG_3.$$

Theorem 5.1. The equilibrium point E_0 is unstable.

Proof:

The system's variational matrix J obtained at the equilibrium point E_0 is,

$$J(0, 0, 0) = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & -m \end{bmatrix}.$$

The eigenvalues are $\lambda_1 = C_1 > 0$, $\lambda_2 = C_2 > 0$ and $\lambda_3 = -m < 0$.

Therefore, the point of equilibrium E_0 is unstable.

Theorem 5.2. If $\Gamma > 1$ and

$$m > \frac{\partial U_1 \Gamma e_1 (C_1 + \mathfrak{P}e_1)}{[C_1^2 \Gamma^2 e_1^2 + S^2 (C_1 + \mathfrak{P}e_1)^2]} + \frac{U_2 d C_2 e_2 (C_2 + n e_2)}{[\gamma (C_2 + n e_2) + C_2 e_2]},$$

the equilibrium point E_1 exhibits stable.

Proof:

The system's variational matrix J obtained at the equilibrium point E_1 is,

$$J(G_1, G_2, 0) = \begin{bmatrix} C_1(1 - 2\Gamma) & 0 & -C_1^2 \omega \Gamma e_1 (1 + \Gamma) - \frac{\partial C_1 \Gamma e_1 (C_1 + \mathfrak{P}e_1)}{[C_1^2 \Gamma^2 e_1^2 + S^2 (C_1 + \mathfrak{P}e_1)^2]} \\ 0 & -C_2 & - \frac{d C_2 e_2 (C_2 + n e_2)}{[\gamma (C_2 + n e_2) + C_2 e_2]} \\ 0 & 0 & \frac{\partial U_1 \Gamma e_1 (C_1 + \mathfrak{P}e_1)}{[C_1^2 \Gamma^2 e_1^2 + S^2 (C_1 + \mathfrak{P}e_1)^2]} + \frac{U_2 d C_2 e_2 (C_2 + n e_2)}{[\gamma (C_2 + n e_2) + C_2 e_2]} - m \end{bmatrix}.$$

The eigenvalues of the above Jacobian matrix are,

$$\lambda_1 = C_1(1 - 2\Gamma),$$

$$\lambda_2 = -C_2,$$

$$\lambda_3 = \frac{\partial U_1 \Gamma e_1 (C_1 + \mathfrak{P}e_1)}{[C_1^2 \Gamma^2 e_1^2 + S^2 (C_1 + \mathfrak{P}e_1)^2]} + \frac{U_2 d C_2 e_2 (C_2 + n e_2)}{[\gamma (C_2 + n e_2) + C_2 e_2]} - m.$$

If $\Gamma > 1$ and

$$m > \frac{\partial U_1 \Gamma e_1 (C_1 + \mathfrak{P}e_1)}{[C_1^2 \Gamma^2 e_1^2 + S^2 (C_1 + \mathfrak{P}e_1)^2]} + \frac{U_2 d C_2 e_2 (C_2 + n e_2)}{[\gamma (C_2 + n e_2) + C_2 e_2]},$$

then the above eigenvalues are negative.

Hence, the equilibrium point E_1 is stable.

Theorem 5.3. If $C_1\Gamma + \frac{C_1(1-\Gamma)}{(1+\omega B^*)} < \frac{\delta B^*}{S^2}$, $tr(J(E_2)) < 0$ and $det(J(E_2)) >$

0, then the equilibrium point E_2 is locally asymptotically stable.

Proof:

The variational matrix J at E_2 is,

$$J(0, G_2, G_3) = \begin{bmatrix} \Lambda_{11} & 0 & 0 \\ 0 & \Lambda_{22} & \Lambda_{23} \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} \end{bmatrix},$$

$$\text{Where } \Lambda_{11} = C_1\Gamma + \frac{C_1(1-\Gamma)}{(1+\omega)B^*} - \frac{\delta B^*}{S^2},$$

$$\Lambda_{12} = 0,$$

$$\Lambda_{13} = 0,$$

$$\Lambda_{21} = 0,$$

$$\Lambda_{22} = C_2 - \frac{2C_2A^*}{e_2} - \frac{dB^*}{(\gamma + A^* + QB^*)} + \frac{dA^*B^*}{(\gamma + A^* + QB^*)^2} - 2nA^*,$$

$$\Lambda_{23} = -\frac{dA^*}{(\gamma + A^* + QB^*)} + \frac{QdA^*B^*}{(\gamma + A^* + QB^*)^2},$$

$$\Lambda_{31} = \frac{\delta U_1 B^*}{S^2},$$

$$\Lambda_{32} = \frac{U_2 dA^*}{(\gamma + A^* + QB^*)} - \frac{U_2 dA^* B^*}{(\gamma + A^* + QB^*)^2},$$

$$\Lambda_{33} = \frac{U_2 dA^*}{(\gamma + A^* + QB^*)} - \frac{QU_2 dA^* B^*}{(\gamma + A^* + QB^*)^2} - m - 2\hat{h}B^*.$$

The characteristic equation of the above matrix is given by:

$$(\lambda - \Lambda_{11})[\lambda^2 - tr(J(E_2))\lambda + det(J(E_2))] = 0.$$

The eigenvalue is

$$\lambda_1 = C_1\Gamma + \frac{C_1(1-\Gamma)}{(1+\omega)B^*} - \frac{\phi B^*}{S^2}$$

and the corresponding Jacobian matrix for the above equation is,

$$J(E_2) = \begin{bmatrix} \Lambda_{22} & \Lambda_{23} \\ \Lambda_{32} & \Lambda_{33} \end{bmatrix},$$

Here

$$tr(J(E_2)) = C_2 - \frac{2C_2A^*}{e_2} - 2(nA^* + hB^*) - m + \frac{d}{(\gamma + A^* + QB^*)} (B^* + U_2A^*)$$

$$- \frac{QdA^*B^*}{(\gamma + A^* + QB^*)^2} [1 + U_2] < 0$$

$$det(J(E_2)) = (C_2 - \frac{2C_2A^*}{e_2} - \frac{dB^*}{(\gamma + A^* + QB^*)} + \frac{dA^*B^*}{(\gamma + A^* + QB^*)^2} - 2nA^*)$$

$$(\frac{U_2dA^*}{(\gamma + A^* + QB^*)} - \frac{QU_2dA^*B^*}{(\gamma + A^* + QB^*)^2} - m - 2hB^*)$$

$$-(-\frac{dA^*}{(\gamma + A^* + QB^*)} + \frac{QdA^*B^*}{(\gamma + A^* + QB^*)^2})$$

$$(\frac{U_2dA^*}{(\gamma + A^* + QB^*)} - \frac{U_2dA^*B^*}{(\gamma + A^* + QB^*)^2})$$

By the Routh-Hurwitz criteria, the equilibrium point E_2 is locally asymptotically stable if

$$C_1\Gamma + \frac{C_1(1-\Gamma)}{(1+\omega)B^*} < \frac{\phi B^*}{S^2}, tr(J(E_2)) < 0 \text{ and}$$

$det(J(E_2)) > 0$. Hence, the proof is completed.

Theorem 5.4. The equilibrium point E_3 is locally asymptotically stable if the following condition are satisfied.

- (1) $\chi_1 > 0$,
- (2) $\chi_2 > 0$,

$$(3) \chi_1 \chi_2 - \chi_3 > 0.$$

Proof:

The system's variational matrix is,

$$J(G_1, G_2, G_3) = \begin{bmatrix} \delta_{11} & 0 & \delta_{13} \\ 0 & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix},$$

$$\text{where } \delta_{11} = C_1 \Gamma + \frac{C_1(1-\Gamma)}{(1+\omega G_3^*)} - \frac{2C_1 G_1^*}{e_1} - \frac{\sigma G_3^*}{(G_1^{*2} + S^2)} + \frac{2\sigma G_1^{*2} G_3^*}{(G_1^{*2} + S^2)^2} - 2\mathfrak{P} G_1^*$$

$$\delta_{12} = 0,$$

$$\delta_{13} = -\frac{C_1 \omega (1-\Gamma) G_1^*}{(1+\omega G_3^*)^2} - \frac{\sigma G_1^*}{(G_1^{*2} + S^2)},$$

$$\delta_{21} = 0,$$

$$\delta_{22} = C_2 - \frac{2C_2 G_2^*}{e_2} - \frac{d G_3^*}{(\gamma + G_2^* + Q G_3^*)} + \frac{d G_2^*}{G_3} (\gamma + G_2^* + Q G_3^*)^2 - 2n G_2^*,$$

$$\delta_{23} = -\frac{d G_2^*}{(\gamma + G_2^* + Q G_3^*)} + \frac{Q d G_2^* G_3^*}{(\gamma + G_2^* + Q G_3^*)^2},$$

$$\delta_{31} = \frac{U_1 \sigma G_3^*}{(G_1^{*2} + S^2)} + \frac{2U_1 \sigma G_1^{*2} G_3^*}{(G_1^{*2} + S^2)^2},$$

$$\delta_{32} = \frac{U_2 d G_3^*}{(\gamma + G_2^* + Q G_3^*)} - \frac{U_2 d G_2^* G_3^*}{(\gamma + G_2^* + Q G_3^*)^2},$$

$$\delta_{33} = \frac{U_1 \sigma G_1^*}{(G_1^{*2} + S^2)} + \frac{U_2 d G_2^*}{(\gamma + G_2^* + Q G_3^*)} - \frac{Q U_2 d G_2^* G_3^*}{(\gamma + G_2^* + Q G_3^*)^2} - m - 2h G_3^*.$$

The characteristic equation is,

$$\lambda^3 + \chi_1 \lambda^2 + \chi_2 \lambda + \chi_3 = 0.$$

$$\chi_1 = -(\delta_{11} + \delta_{22} + \delta_{33})$$

$$= -C_1 \Gamma - \frac{C_1(1-\Gamma)}{(1+\omega G_3^*)} + \frac{2C_1 G_1^*}{e_1} - \frac{\sigma G_3^*}{(G_1^{*2} + S^2)} - \frac{2\sigma G_1^{*2} G_3^*}{(G_1^{*2} + S^2)^2} + 2\mathfrak{P} G_1^* - C_2$$

$$+ \frac{2C_2G_2^*}{e_2} + \frac{dG_3^*}{(\gamma + G_2^* + QG_3^*)} - \frac{dG_2^*}{G_3} (\gamma + G_2^* + QG_3^*)^2 + 2nG_2^*$$

$$- \frac{U_1\phi G_1^*}{(G_1^{*2} + S^2)} - \frac{U_2dG_2^*}{(\gamma + G_2^* + QG_3^*)} + \frac{QU_2dG_2^*G_3^*}{(\gamma + G_2^* + QG_3^*)^2} + m + 2hG_3^*.$$

$$\chi_2 = \delta_{11}(\delta_{22} + \delta_{33}) + \delta_{22}\delta_{33} + \delta_{13}\delta_{31} - \delta_{23}\delta_{32}$$

$$= (C_1\Gamma + \frac{C_1(1-\Gamma)}{(1+\omega G_3^*)} - \frac{2C_1G_1^*}{e_1} - \frac{\phi G_3^*}{(G_1^{*2} + S^2)} + \frac{2\phi G_1^{*2}G_3^*}{(G_1^{*2} + S^2)^2} - 2\mathfrak{P}G_1^*)$$

$$(C_2 - \frac{2C_2G_2^*}{e_2} - \frac{dG_3^*}{(\gamma + G_2^* + QG_3^*)} + \frac{dG_2^*}{G_3} (\gamma + G_2^* + QG_3^*)^2 - 2nG_2^*)$$

$$+ (\frac{U_1\phi G_1^*}{(G_1^{*2} + S^2)} + \frac{U_2dG_2^*}{(\gamma + G_2^* + QG_3^*)} - \frac{QU_2dG_2^*G_3^*}{(\gamma + G_2^* + QG_3^*)^2} - m - 2hG_3^*)$$

$$+ (C_2 - \frac{2C_2G_2^*}{e_2} - \frac{dG_3^*}{(\gamma + G_2^* + QG_3^*)} + \frac{dG_2^*}{G_3} (\gamma + G_2^* + QG_3^*)^2 - 2nG_2^*)$$

$$(\frac{U_1\phi G_1^*}{(G_1^{*2} + S^2)} + \frac{U_2dG_2^*}{(\gamma + G_2^* + QG_3^*)} - \frac{QU_2dG_2^*G_3^*}{(\gamma + G_2^* + QG_3^*)^2} - m - 2hG_3^*)$$

$$+ (-\frac{C_1\omega(1-\Gamma)G_1^*}{(1+\omega G_3^*)^2} - \frac{\phi G_1^*}{(G_1^{*2} + S^2)}) (\frac{U_1\phi G_3^*}{(G_1^{*2} + S^2)} + \frac{2U_1\phi G_1^{*2}G_3^*}{(G_1^{*2} + S^2)^2})$$

$$- (-\frac{dG_2^*}{(\gamma + G_2^* + QG_3^*)} + \frac{QdG_2^*G_3^*}{(\gamma + G_2^* + QG_3^*)^2})$$

$$(\frac{U_2dG_3^*}{(\gamma + G_2^* + QG_3^*)} - \frac{U_2dG_2^*G_3^*}{(\gamma + G_2^* + QG_3^*)^2})$$

$$\chi_3 = \delta_{11}\delta_{22}\delta_{33} - \delta_{11}\delta_{23}\delta_{32} + \delta_{13}\delta_{22}\delta_{31}$$

$$\begin{aligned}
&= \left(C_1 \Gamma + \frac{C_1(1-\Gamma)}{(1+\omega G_3^*)} - \frac{2C_1 G_1^*}{e_1} - \frac{\delta G_3^*}{(G_1^{*2} + S^2)} + \frac{2\delta G_1^{*2} G_3^*}{(G_1^{*2} + S^2)^2} - 2\mathfrak{P} G_1^* \right) \\
&\quad \left(C_2 - \frac{2C_2 G_2^*}{e_2} - \frac{d G_3^*}{(\gamma + G_2^* + Q G_3^*)} + \frac{d G_2^*}{G_3} (\gamma + G_2^* + Q G_3)^{*2} - 2n G_2^* \right) \\
&\quad \left(\frac{U_1 \delta G_1^*}{(G_1^{*2} + S^2)} + \frac{U_2 d G_2^*}{(\gamma + G_2^* + Q G_3^*)} - \frac{Q U_2 d G_2^* G_3^*}{(\gamma + G_2^* + Q G_3)^{*2}} - m - 2\hat{h} G_3^* \right) \\
&\quad - \left(C_1 \Gamma + \frac{C_1(1-\Gamma)}{(1+\omega G_3^*)} - \frac{2C_1 G_1^*}{e_1} - \frac{\delta G_3^*}{(G_1^{*2} + S^2)} + \frac{2\delta G_1^{*2} G_3^*}{(G_1^{*2} + S^2)^2} - 2\mathfrak{P} G_1^* \right) \\
&\quad \left(-\frac{d G_2^*}{(\gamma + G_2^* + Q G_3^*)} + \frac{Q d G_2^* G_3^*}{(\gamma + G_2^* + Q G_3)^{*2}} \right) \\
&\quad \left(\frac{U_2 d G_3^*}{(\gamma + G_2^* + Q G_3^*)} - \frac{U_2 d G_2^* G_3^*}{(\gamma + G_2^* + Q G_3)^{*2}} \right) + \left(-\frac{C_1 \omega (1-\Gamma) G_1^*}{(1+\omega G_3^*)^2} - \frac{\delta G_1^*}{(G_1^{*2} + S^2)} \right) \\
&\quad \left(C_2 - \frac{2C_2 G_2^*}{e_2} - \frac{d G_3^*}{(\gamma + G_2^* + Q G_3^*)} + \frac{d G_2^*}{G_3} (\gamma + G_2^* + Q G_3)^{*2} - 2n G_2^* \right) \\
&\quad \left(\frac{U_1 \delta G_3^*}{(G_1^{*2} + S^2)} + \frac{2U_1 \delta G_1^{*2} G_3^*}{(G_1^{*2} + S^2)^2} \right)
\end{aligned}$$

By applying the Routh-Hurwitz criterion and the stability of the coexistence equilibrium point E_3 exhibits locally asymptotically stability if $\chi_1 > 0$, $\chi_2 > 0$ and $\chi_1 \chi_2 - \chi_3 > 0$ are satisfied.

Hence, the theorem is proved.

6 SADDLE NODE BIFURCATION ANALYSIS

Sotomayor's theory was applied to identify the stable states that exhibit bifurcation conditions. Numerous researchers have used Sotomayor's theory to examine various bifurcation. Moreover, model (2.1) can be modified as described below:

$$\frac{dG}{dt} = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} G_1 f_1(G_1, G_2, G_3) \\ G_2 f_2(G_1, G_2, G_3) \end{bmatrix}$$

$$dt \text{ with } G = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} \text{ and } F(G_1, G_2, G_3) = \begin{bmatrix} G_1 f_1(G_1, G_2, G_3) \\ G_2 f_2(G_1, G_2, G_3) \\ G_3 f_3(G_1, G_2, G_3) \end{bmatrix}$$

Next, the second order derivative of $F(G_1, G_2, G_3)$ with respect to G is given by:

$$D^2F(G_1, G_2, G_3) = \begin{bmatrix} -\frac{2C_1}{e} + \frac{6\delta G_1 G_3}{(G^2 + S^2)} - \frac{8\delta G_1^5 G_3}{(G^2 + S^2)^4} - 2\mathfrak{P} \\ -\frac{2C_2}{e_2} + \frac{2dG_3}{(\gamma + G_2 + QG_3)^2} - \frac{2dG_2^2 G_3}{(\gamma + G_2 + QG_3)^4} - 2n \\ -\frac{2QdU_2 G_2}{(\gamma + G_2 + QG_3)^2} + \frac{2Q^3 U_2 dG_2 G_3^2}{(\gamma + G_2 + QG_3)^4} - 2\hat{h} \end{bmatrix}.$$

Theorem 6.1. When Γ passes through $\Gamma^* = \frac{1}{C_1} \left[\frac{\delta B^*}{S^2} - \frac{C_1(1 - \Gamma)}{(1 + \omega B^*)} \right]$,

system (2.1) experience a saddle node bifurcation at E_2 if $tr(J(E_2)) < 0$ and $det(J(E_2)) > 0$ holds.

Proof:

The eigenvalue λ_1 of $J(0, G_2, G_3)$ becomes zero when $\Gamma = \Gamma^*$ and negative under Routh-Hurwitz criteria $tr(J(E_2)) < 0$ and $det(J(E_2)) > 0$.

Let $\mathfrak{B}_1$ and $\mathcal{L}_1$ be eigenvectors of $J(0, G_2, G_3)$ and $J^T(0, G_2, G_3)$ and the corresponding eigenvalue $\lambda_1 = 0$, respectively. Here,

$$\mathfrak{B}_1 = \begin{bmatrix} 0 \\ 1 \\ \frac{\Lambda_{22}}{\Lambda_{23}} \end{bmatrix} \text{ and } \mathcal{L}_1 = \begin{bmatrix} -\frac{\Lambda_{31}}{\Lambda_{11}} \\ -\frac{\Lambda_{32}}{\Lambda_{22}} \\ 1 \end{bmatrix}$$

Straight computation gives the following equation:

$$F_{\Gamma^*}(E_2, \Gamma^*) = \begin{bmatrix} -\frac{1}{(1 + \omega B^*)} \\ 0 \\ 0 \end{bmatrix}.$$

$$\mathcal{L}_1^T F_{\Gamma^*}(E_2, \Gamma^*) = \frac{\Lambda_{31}}{\Lambda_{11}(1 + \omega B^*)} \neq 0.$$

$$\mathcal{L}_1^T [D^2 F_{\Gamma^*}(E_2, \Gamma^*)(\mathfrak{B}_1, \mathfrak{B}_1)] = 2 \left(1 + \left(\frac{\Lambda_{22}}{\Lambda_{23}} \right)^2 \right) \left[q \frac{\Lambda_{31}}{\Lambda_{11}} + Q \frac{\Lambda_{32}}{\Lambda_{33}} - \mathfrak{Q} \right] \neq 0.$$

Where,

$$q = \frac{C_1}{e_1} + \mathfrak{P}$$

$$Q = \frac{C_2}{e_2} - \frac{2dB^*}{(\gamma + A^* + QB^*)^2} + \frac{2dA^{*2}B^*}{(\gamma + A^* + QB^*)^4} + n$$

$$\mathfrak{Q} = \frac{2QdU_2A^*}{(\gamma + A^* + QB^*)^2} - \frac{2Q^3U_2dA^*B^{*2}}{(\gamma + A^* + QB^*)^4} + \mathfrak{h}$$

Accordingly, by Sotomayor theorem, system (2.1) near E_2 with $\Gamma = \Gamma^*$ experience the saddle node bifurcation. The proof is completed.

Theorem 6.2. When C_1 passes through

$C_1^* = \frac{1}{\Gamma} [\delta_{11} + \delta_{22} + \delta_{33}]$, system (2.1) experience a saddle node bifurcation at E_3 if $\chi_2 > 0$ and $\chi_1\chi_2 - \chi_3 > 0$ holds.

Proof:

The eigenvalue χ_1 of $J(G_1^*, G_2^*, G_3^*)$ becomes zero, when $C_1 = C_1^*$ and others are negative under Routh-Hurwitz criteria $\chi_2 > 0$ and $\chi_1\chi_2 - \chi_3 > 0$.

Let B_2 and L_2 are eigenvectors of $J(G_1^*, G_2^*, G_3^*)$ and $J^T(G_1^*, G_2^*, G_3^*)$

and the corresponding eigenvalue $\chi_1 = 0$, respectively.

Here,

$$\mathfrak{B}_2 = \begin{bmatrix} \frac{\delta_{12}\delta_{23} - \delta_{12}\delta_{22}}{\delta_{11}\delta_{22}} \\ -\frac{\delta_{23}}{\delta_{22}} \\ 1 \end{bmatrix} \text{ and } \mathcal{L}_1 = \begin{bmatrix} \frac{1}{\delta_{32}\delta_{11} - \delta_{12}\delta_{31}} \\ \frac{\delta_{22}\delta_{31}}{\delta_{11}} \\ -\frac{\delta_{11}}{\delta_{31}} \end{bmatrix}$$

Straight computation gives the following equation:

$$\bar{F}_{\Gamma^*}(E_2, \Gamma^*) = \begin{bmatrix} G_1^* \left(\Gamma + \frac{1 - \Gamma}{(1 + \bar{\omega}G_3^*)} - \frac{G_1^*}{e_1} \right) \\ 0 \\ 0 \end{bmatrix}$$

$$\mathcal{L}_2^T \bar{F}_{\bar{C}_1^*}(E_3, \bar{C}_1^*) = G_1^* \left(\Gamma + \frac{1 - \Gamma}{(1 + \bar{\omega}G_3^*)} - \frac{G_1^*}{e_1} \right) \neq 0.$$

$$\mathcal{L}_2^T[D^2\bar{F}_{\bar{C}_1^*}(E_3, \bar{C}_1^*)(\mathfrak{B}_2, \mathfrak{B}_2)] = R \left[\mathcal{H}_1 + \mathcal{H}_2 \frac{\delta_{32}\delta_{11} - \delta_{12}\delta_{31}}{\delta_{22}\delta_{31}} - \mathcal{H}_3 \frac{\delta_{23}}{\delta_{22}} \right] \neq 0.$$

Where,

$$R = \left(\frac{\delta_{12}\delta_{23} - \delta_{12}\delta_{22}}{\delta_{11}\delta_{22}} \right)^2 + \left(\frac{\delta_{23}}{\delta_{22}} \right)^2 + 1$$

$$\mathcal{H}_1 = -\frac{2\bar{C}_1^*}{e_1} + \frac{6\delta G_1^* G_3^*}{(G_1^{*2} + \bar{S}^2)} - \frac{8\delta G_1^{*5} G_3^*}{(G_1^{*2} + \bar{S}^2)^4} - 2\mathfrak{P}$$

$$\mathcal{H}_2 = -\frac{2\bar{C}_2}{e_2} + \frac{2\bar{d}G_3^*}{(\gamma + G_2^* + QG_3^*)^2} - \frac{2\bar{d}G_2^{*2}G_3^*}{(\gamma + G_2^* + QG_3^*)^4} - 2n$$

$$\mathcal{H}_3 = -\frac{2Q\bar{d}U_2G_2^*}{(\gamma + G_2^* + QG_3^*)^2} + \frac{2Q^3U_2\bar{d}G_2^*G_3^{*2}}{(\gamma + G_2^* + QG_3^*)^4} - 2\hat{h}$$

Accordingly, by Sotomayor theorem, system (2.1) near E_2 with $\Gamma = \Gamma^*$ experience the saddle node bifurcation.

The proof is completed.

7 NUMERICAL SIMULATIONS

To illustrate the dynamical behaviour of the proposed model, the following numerical simulations are carried out using MATLAB to generate the corresponding figures. The parameter values used in the simulations are assumed values selected to demonstrate the population dynamics, stability properties and bifurcation behaviour of the model.

The first set of assumed parameter values is $C_1^- = 1.7725$, $C_2^- = 2.5$, $e_1 = 8$, $e_2 = 9$, $\Gamma = 0.5$, $\bar{\omega} = 0.4$, $\hat{o} = 1$, $S^- = 1$, $P = 0.1$, $d^- = 1$, $\gamma = 1$, $Q = 0.5$, $n = 0.1$, $U_1 = 0.46$, $U_2 = 0.51$, $m = 0.48$ and $h^* = 0.1$, with initial populations $G_1(0) = 1.2$, $G_2(0) = 2.5$ and $G_3(0) = 5.4$. The resulting trajectories of $G_1(t)$, $G_2(t)$ and $G_3(t)$ illustrate the interaction among the two prey populations and the predator population, as shown in Figure 1.

For the stability of the coexistence equilibrium. $E_3 = (G_1^*, G_2^*, G_3^*)$, the assumed parameter values are $C_1^- = 0.26562638$, $C_2^- = 2.09202627$, $e_1 =$

10.41019572, $e_2 = 7.49297222$, $\Gamma = 0.25647719$, $\bar{\omega} = 0.54650155$, $\hat{o} = 1.21857695$, $S^- = 0.65973891$, $P = 0.15222389$, $d^- = 0.67050803$, $\gamma = 0.89122819$, $Q = 0.61339215$, $n = 0.10320676$, $U_1 = 0.59339929$, $U_2 = 0.66891889$, $m = 0.67594699$ and $h^* = 0.07831420$. The corresponding trajectories approach the positive coexistence equilibrium E_3 , indicating the coexistence of the three populations and supporting the stability of E_3 , as shown in Figure 2.

The saddle-node bifurcation associated with the equilibrium E_2 is investigated using the assumed parameter values $C_1^- = 1.7725$, $C_2^- = 2.5$, $e_1 = 8$, $e_2 = 9$, $\bar{\omega} = 0.4$, $\hat{o} =$

1, $S^- = 0.99$, $P = 0.1$, $d^- = 1$, $\gamma = 0.7$, $Q = 0.66$, $n = 0.1$, $U_1 = 0.46$, $U_2 = 0.60$, $m = 0.40$ and $h^* = 0.05$. The bifurcation diagram is obtained by varying Γ . The equilibrium E_2 is stable for a certain range of Γ and loses its stability as Γ passes through the critical value. Since the saddle-node bifurcation conditions are satisfied at the critical value of Γ , the numerical results confirm the occurrence of a saddle-node bifurcation at E_2 , as shown in Figure 3.

The saddle-node bifurcation associated with the coexistence equilibrium E_3 is examined using the assumed parameter values $C_2^- = 2.5$, $e_1 = 6$, $e_2 = 5$, $\Gamma = 0.5$, $\bar{\omega} = 0.4$, $\hat{o} = 0.99$, $S^- = 1$, $P = 0.2$, $d^- = 1$, $\gamma = 0.98$, $Q = 0.5$, $n = 0.1$, $U_1 = 0.46$, $U_2 = 0.51$, $m = 0.48$ and $h^* = 0.1$. The bifurcation diagram is obtained by varying C_1^- . The equilibrium E_3 is stable over a certain range of C_1^- and undergoes a change in stability as C_1^- passes through its critical value. Since the saddle-node bifurcation conditions are satisfied at the critical value of C_1^- , the numerical results confirm the occurrence of a saddle-node bifurcation at E_3 , as shown in Figure 4.

The influence of intraspecific competition on the first prey population $G_1(t)$ is investigated by varying the parameter P , while keeping the other parameters fixed. The term $-\mathfrak{P}G_1^2$ represents the intraspecific competition among individuals of G_1 . Increasing P strengthens the competition within the first prey population and produces a stronger density-dependent reduction in $G_1(t)$. The effect of P on $G_1(t)$ is shown in Figure 5.

Similarly, the effect of intraspecific competition on the second prey population $G_2(t)$ is studied by varying n , while keeping the remaining parameters fixed. The term involving $-nG_2^2$ represents the intraspecific competition within G_2 . An increase in n strengthens the competition among individuals of the second prey population and consequently affects its population density. The influence of n on $G_2(t)$ is shown in Figure 6.

Finally, the effect of intraspecific competition on the predator population $G_3(t)$ is examined by varying h^* , while keeping the other parameters unchanged. The parameter h^* represents the intraspecific competition effect within the predator population. Increasing h^* strengthens the competition among predator individuals and influences the population density of $G_3(t)$. The effect of h^* on $G_3(t)$ is shown in Figure 7.

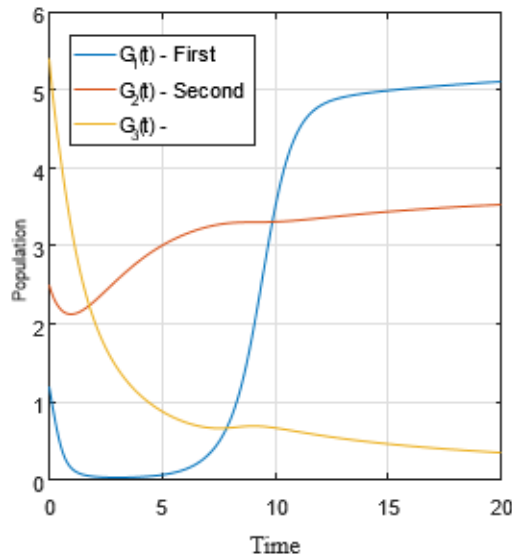


Figure 1: Dynamics of the System.

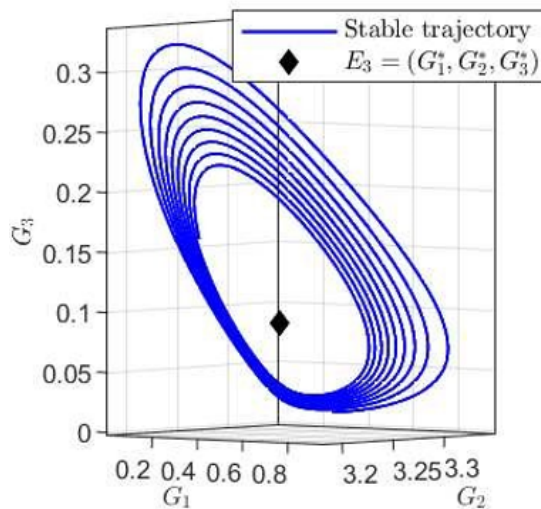


Figure 2: Equilibrium Point E_3 is stable.

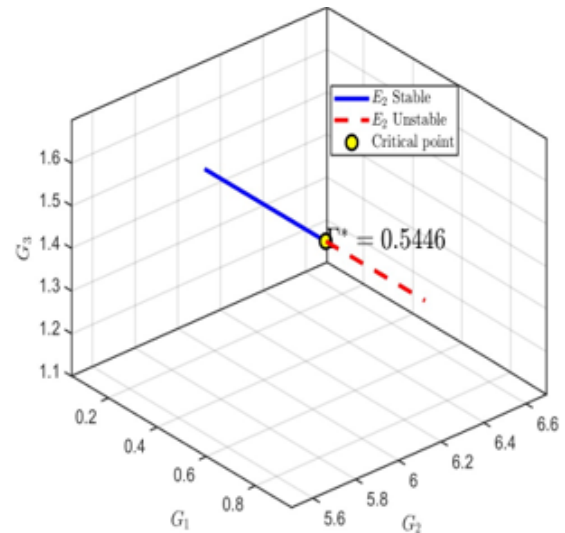


Figure 3: Saddle-Node Bifurcation at E_2 .

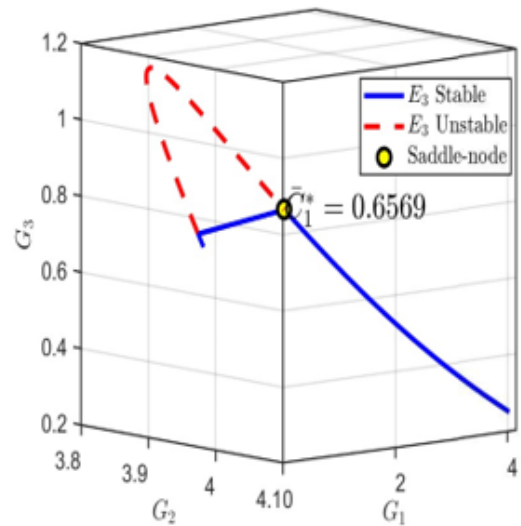


Figure 4: Saddle-Node Bifurcation at E_3 ..

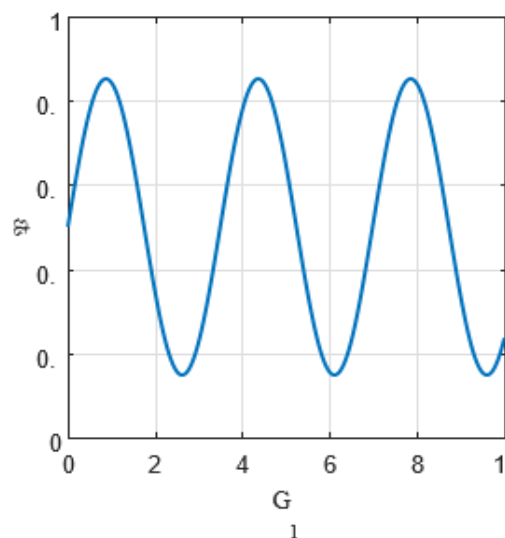


Figure 5: Intraspecific Competition on $G_1(t)$.

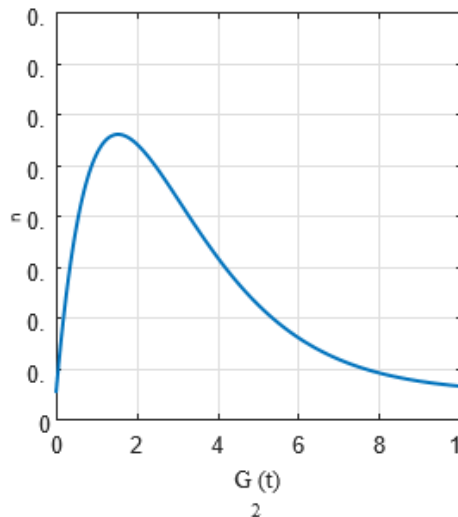


Figure 6: Intraspecific Competition on $G_2(t)$.

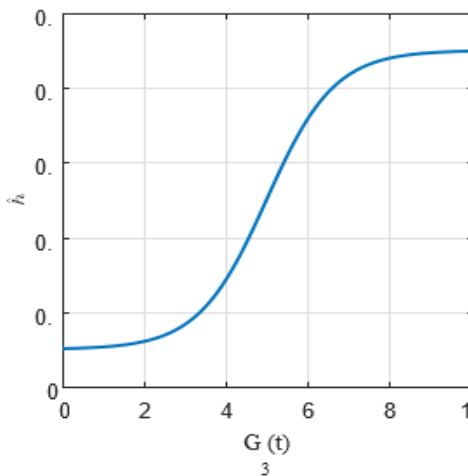


Figure 7: Intraspecific Competition on $G_3(t)$.

8 CONCLUSION

This article investigates a three dimensional prey-predator model involving two prey populations and one predator population, incorporating saturated fear effects, intraspecific competition and different functional responses. The Holling type IV functional response is considered for the first prey population, while the Beddington-DeAngelis functional response is used for the second prey population. The model is shown to possess positive and bounded solutions, which ensures the biological feasibility of the populations. The local stability of the coexistence equilibrium is analysed using the Jacobian matrix and the corresponding stability conditions.

The bifurcation analysis shows that the equilibria E_2 and E_3 undergo saddle-node bifurcations when the corresponding critical parameter values are reached. In particular, the parameter Γ is considered for the saddle node bifurcation associated with E_2 , while C^{-1} is considered for the saddle node bifurcation associated with E_3 . The numerical simulations support the analytical results and illustrate the changes in stability near the critical parameter values.

The influence of intraspecific competition is also examined for all three populations. The parameter P describes the intraspecific competition in the first prey population G_1 , n represents the corresponding competition effect in the second prey population G_2 , and h represents the intraspecific competition effect in the predator population G_3 . The numerical results demonstrate that changes in these parameters can significantly influence the population levels and the interaction among the species.

A practical example of the proposed model can be found in agricultural ecosystems, where two pest species feed on crops and share a common natural predator such as a ladybird beetle, lacewing or predatory bug. The two pest populations may compete for the same food resources, while the predator feeds on both pest species. The presence of predators can also produce fear effects that modify the feeding, movement and activity of the pest populations. In addition, competition among individuals of the same pest or predator species becomes stronger as their population densities increase. Thus, the proposed model can help describe how changes in fear effects, intraspecific competition and predator feeding behaviour influence the persistence and coexistence of pest and predator populations. Overall, the results provide useful mathematical insight into the stability and bifurcation behaviour of interacting populations in ecological systems.

REFERENCES

- [1] N. Ali, (2026), A Prey-Predator Model with Holling Type-IV Functional Response, *International Journal of Research on Social and Natural Sciences*, **1(1)**.
- [2] L. Cai, J. Yu and G. Zhu, (2008), A stage-structured predator-prey model with Beddington-DeAngelis functional response, *Journal of Applied Mathematics and Computing*, **26**, 85–103.
- [3] D. Das1, P. Bora, S. Mondal, S. Borah and P. Gogoi, (2022), An Analysis Of A Two Predator One Prey Model With Predator Advantage, *Journal of Pharmaceutical Negative Results*, **13(5)**, 1679–1689.
- [4] W. Gao and B. Dai, (2019), Dynamics of a Predator-Prey Model with Delay and Fear Effect, *Journal of Nonlinear Modelling and Analysis*, **1(1)**, 57–72.
- [5] G. He, F. Chen and Z. Li, (2025), The Influence of Saturated Fear Effects on Species Dynamics: A Lotka-Volterra Competition Model Analysis, *International Journal of Biomathematics*, **33(10)**.
- [6] W. Ishaque, Q. Din and K. A. Khan, (2024), Dynamics of Predator-Prey Model Based on Fear Effect with Bifurcation Analysis and Chaos Control, *Qualitative Theory of Dynamical Systems*, **23(1)**.
- [7] P. Kalra and R. Kaur, (2020), A Prey-Predator Model with Beddington-De Angelis Holling type IV Functional Response, *Sematic Scholar*, **6(1)**, 452–457.

- [8] S. Pal, S. Majhi, S. Mandal, (2019), Role of Fear in a Predator-Prey Model with Beddington-DeAngelis Functional Response, *Zeitschrift Für Naturforschung A*, **74(7)**, 581–595.
- [9] S. Pal, F. Al Basir and S. Ray, (2023), Impact of Cooperation and IntraSpecific Competition of Prey on the Stability of Prey-Predator Models with Refuge, *Mathematical and Computational Applications*, **28(4)**.
- [10] H. S. Panigoro, E. Rahmi E and R. Resmawan, (2023), Bifurcation Analysis of a Predator Prey Model Involving Age Structure, Intraspecific Competition, Michaelis Menten Type Harvesting, and Memory Effect, *Frontiers in Applied Mathematics and Statistics*, **8**, 1–14.
- [11] P. Panja, (2021), Dynamics of a Predator-prey Model with Beddington DeAngelis Functional Response, Omnivory and Predator Switching, *Journal of Vibration Testing and System Dynamics*, **5(4)**, 407–428.[
- [12] D. Sen, S. Ghorai, M. Banerjee and A. Morozov, (2021), Bifurcation Analysis of the Predator-Prey Model with the Allee Effect in the Predator, *Journal of Mathematical Biology*, **84(1-2)**, 1–27.
- [13] A. Sha and J. Chattopadhyay, (2023), Dynamical Study Of Fear Effect in Prey-Predator Model with Disease in Predator, *Journal of Biological Systems*, **31(4)**, 1319–1340.
- [14] Y. Shao and W. Kong, (2022), A Predator-Prey Model with Beddington DeAngelis Functional Response and Multiple Delays in Deterministic and Stochastic Environments, *Mathematics*, **10(18)**.
- [15] D. Wang and Y. Gao, (2025), Prey Predator Model with Beddington DeAngelis Functional Response, *Proceedings of the 2024 International Conference on Mathematics and Machine Learning*, **12**, 127–131.
- [16] X. Wang, L. Zanette and X. Zou, 2016, Modelling the Fear Effect in Predator Prey Interactions, *Journal of Mathematical Biology*, **73**, 1179– 1204.
- [17] J. Yu, F. Sun and Y. Chao, (2020), Study of a Predator-Prey Model with Holling Type IV Functional Response, *International Conference on Control, Automation and Robotics (ICCAR), IEEE*.
- [18] H. Zhu and W. Zhou, (2025), Stability Analysis of a Predator-Prey Model with Saturated Fear Cost Effect, *Journal of Physics Conference Series*, **3213(1)**, 1–7.