RESEARCH PAPER

Topological Structures of Cubic Bipolar Intuitionistic Fuzzy Sets

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ABSTRACT
Cubic bipolar intuitionistic fuzzy sets (CBIFS) is a novel hybrid extension of the bipolar intuitionistic fuzzy sets (BIFS). A CBIFS is a powerful model for handling fuzziness and bipolarity in terms of positive and negative membership grades, as well as positive and negative non-membership grades. This study aims to define the cubic bipolar intuitionistic fuzzy topology under P-order (PCBIFS topology) and the cubic bipolar intuitionistic fuzzy topology under R-order (RCBIFS topology). In comparison to certain existing techniques, we examine specific features and outcomes of PCBIFS and RCBIFS topologies.
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1. INTRODUCTION

In MCDM, several people choose at the same time from the possibilities in front of them. Most decisions made by groups are distinct from those made by a single person. However, because of the environment's growing complexity in choice analysis and the circumstances themselves, decision makers are occasionally persuaded by numerical information to provide decision making information. As a result, Atanassov [2] demonstrated the intuitionistic fuzzy set (IFS) which has both membership degree and non-membership degree, by extending the fuzzy set (FS) introduced by Zadeh [13] whose members only have a membership grade. The IFS was further extended by Atanassov and Gargov [3] and proposed the notion of interval-valued intuitionistic fuzzy sets (IVIFS), and outlined the fundamental IVIFS algorithm.

Inevitably, both IFS and IVIFS successfully addresses the real-world problems that allow for ambiguity; nonetheless, further research is necessary to find a substitute that both possesses IFS and IVIFS. In order to cope with this problem, Jun et.al. [5] created the idea of a Cubic set and its related operations, which is an extension of a fuzzy set and interval-valued fuzzy set. The author's analysis in the cubic fuzzy environment has solely taken into account the degree of acceptance region, leaving out the degree of rejection. However, it is evident from our daily decisions that the level of rejection is a major factor in performance reviews. In light of these characteristics, Kaur and Garg [6] expanded the idea of the CFS to create the cubic Intuitionistic Fuzzy Sets (CIFS), where the degree of rejection was also included in the analysis.

With the goal of improving the domain of membership degrees in fuzzy sets, Zhang [14, 15] presented a description of a entirely novel extended fuzzy set known as Bipolar Fuzzy Set (BFS). It is symbolized by two elements: one is linked to the grade of positive membership degree which goes over the range from 0 to 1, while the other is linked to the grade of negative membership degree which goes over from -1 to 0. The positive aspects of BFS illustrate what is feasible or permitted, while the negative aspects highlight what is impractical or prohibited. Wei et al. [12] introduced the idea of interval-valued bipolar fuzzy set (IVBFS) in order to broaden the scope of BFS. Four distinct components make up the concept of a bipolar intuitionistic fuzzy sets (BIFS), which Ezhil et al. [4] put forward as an extension of bipolar fuzzy set. The positive membership and the positive non-membership degree falls within the range from 0 to 1, whereas the negative membership and the negative non-membership degrees falls within the

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range from -1 to 0.

In order to address a wide range of challenging issues, the researchers have effectively implemented numerous optimization strategies. When it comes to accurately describing the occurrence of evaluations or assessments, the concept of a simple BIFS falls short due to its limited informational scope and lack of capacity to capture the occurrence of ambiguity and uncertainty, particularly in situations where decision-making involves delicate cases. In a similar manner, the notion of an interval-valued bipolar intuitionistic fuzzy set is likewise inadequate for revealing the experts opinions that depend on the characteristics of the options. In order to do this, we presented a unique model known as Cubic Bipolar Intuitionistic Fuzzy Sets (CBIFS) [11].

Cubic topology was first introduced by Akhter et al [1]. The notion of cubic intuitionistic fuzzy topology was proposed by Riaz et al [9]. The idea of bipolar fuzzy topological spaces was introduced by Kim et al [7]. Ludi Jancy Jenifer and Helen [8] introduced bipolar intuitionistic fuzzy topology. Later, Riaz et al [10] introduced the notion of cubic bipolar fuzzy topology.

This article's remaining content is organized as followed: Basic definitions and preliminaries are defined in Section 1. Section 2 outlines the P-order CBIF topology and its properties. The concepts of R-order CBIF topology and its topological properties were examined in Section 3. Furthermore, a conclusion will be provided in Section 4.

2. CUBIC BIPOLAR INTUITIONISTIC FUZZY SET

This Section goes over some fundamental definitions and basic operations that are used throughout the rest of the paper. Consider $\mathcal{X}$ to be the universal non-empty set.

Definition 2.1. [11] Let I[0,1] and I*$[-1,0]$ be the set of all closed subintervals of [0, 1] and [-1, 0] respectively. A Cubic Bipolar Intuitionistic Fuzzy Set (CBIFS) on $\mathcal{X}$ can be defined as $\tilde{\mathcal{C}}^I = \{(x, \widetilde{\mathcal{B}}^I(x), \mathcal{B}^I(x))\}: x \in \mathcal{X}\}$ where $\widetilde{\mathcal{B}}^I(x) = \{x, ([\mu^{+l}_{\mathcal{B}^I}(x), \mu^{+u}_{\mathcal{B}^I}(x)], [\mu^{-l}_{\mathcal{B}^I}(x), \mu^{-u}_{\mathcal{B}^I}(x)], [\vartheta^{+l}_{\mathcal{B}^I}(x),$

$\vartheta^{+u}_{\mathcal{B}^I}(x),], [\vartheta^{-l}_{\mathcal{B}^I}(x), \vartheta^{-u}_{\mathcal{B}^I}(x)]): x \in \mathcal{X}\}$ be an interval-valued bipolar intuitionistic fuzzy set (IVBIFS) and $\mathcal{B}^I(x) = \{x, (\mu^{+}_{\mathcal{B}^I}(x), \mu^{-}_{\mathcal{B}^I}(x), \vartheta^{+}_{\mathcal{B}^I}(x), \vartheta^{-}_{\mathcal{B}^I}\ (x)): x \in \mathcal{X}\}$ be a bipolar intuitionistic fuzzy set (BIFS). Generally, a CBIFS can be written as

$\tilde{\mathcal{C}}^I = \{x, ([\mu^{+l}_{\mathcal{B}^I}(x), \mu^{+u}_{\mathcal{B}^I}(x)], [\mu^{-l}_{\mathcal{B}^I}(x), \mu^{-l}_{\mathcal{B}^I}(x)], [\vartheta^{+l}_{\mathcal{B}^I}(x), \vartheta^{+u}_{\mathcal{B}^I}(x)], [\vartheta^{-l}_{\mathcal{B}^I}(x), \vartheta^{-u}_{\mathcal{B}^I}(x)]), (\mu^{+}_{\mathcal{B}^I}(x),$

$\mu^{-}_{\mathcal{B}^I}(x), \vartheta^{+}_{\mathcal{B}^I}(x), \vartheta^{-}_{\mathcal{B}^I}\ (x)): x \in \mathcal{X}\}.$

A cubic bipolar intuitionistic fuzzy elements (CBIFE) can be simply written as

$\tilde{\mathcal{C}}^I = \{([\mu^{+l}_{\mathcal{B}^I}, \mu^{+u}_{\mathcal{B}^I}], [\mu^{-l}_{\mathcal{B}^I}, \mu^{-u}_{\mathcal{B}^I}], [\vartheta^{+l}_{\mathcal{B}^I}, \vartheta^{+u}_{\mathcal{B}^I}], [\vartheta^{-l}_{\mathcal{B}^I}, \vartheta^{-u}_{\mathcal{B}^I}]), (\mu^{+}_{\mathcal{B}^I}, \mu^{-}_{\mathcal{B}^I}, \vartheta^{+}_{\mathcal{B}^I}, \vartheta^{-}_{\mathcal{B}^I})\}.$

Example 2.2[11] Let $\mathcal{X} = \{x_1, x_2, x_3\}$ be an universal set. Then the CBIFS can be given as follows:

$\tilde{\mathcal{C}}^I = \{(x_1, ([0.25, 0.35], [-0.47, -0.39], [0.53, 0.58], [-0.21, -0.15]), (0.23, -0.52, 0.64,$
$-0.27)), (x_2, ([0.43, 0.49], [-0.34, -0.28], [0.10, 0.23], [-0.35, -0.25]), (0.46, -0.31,$

$0.17, -0.30)), (x_3, ([0.56, 0.64], [-0.38, -0.30], [0.20, 0.33], [-0.52, -0.45]), (0.60,$
$-0.36, 0.26, -0.50))\}$

2.1 Basic operations on CBIFS

Definition 2.3. [11] Let $\tilde{\mathcal{C}}^I_1$ and $\tilde{\mathcal{C}}^I_2$ be two cubic bipolar intuitionistic fuzzy sets on $\mathcal{X}$. We say $\tilde{\mathcal{C}}^I_1$ and $\tilde{\mathcal{C}}^I_2$ are equal if:

(i) $[\mu^{+l}_{\mathcal{B}^I_1}(x), \mu^{+u}_{\mathcal{B}^I_1}(x)] = [\mu^{+l}_{\mathcal{B}^I_2}(x), \mu^{+u}_{\mathcal{B}^I_2}(x)]$ (ii) $[\mu^{-l}_{\mathcal{B}^I_1}(x), \mu^{-u}_{\mathcal{B}^I_1}(x)] = [\mu^{-l}_{\mathcal{B}^I_2}(x), \mu^{-u}_{\mathcal{B}^I_2}(x)]$

(iii) $[\vartheta^{+l}_{\mathcal{B}^I_1}(x), \vartheta^{+u}_{\mathcal{B}^I_1}(x)] = [\vartheta^{+l}_{\mathcal{B}^I_2}(x), \vartheta^{+u}_{\mathcal{B}^I_2}(x)]$ (iv) $[\vartheta^{-l}_{\mathcal{B}^I_1}(x), \vartheta^{-u}_{\mathcal{B}^I_1}(x)] = [\vartheta^{-l}_{\mathcal{B}^I_2}(x), \vartheta^{-u}_{\mathcal{B}^I_2}(x)]$

(v) $\mu^{+}_{\mathcal{B}^I_1}(x) = \mu^{+}_{\mathcal{B}^I_2}(x)$ and $\mu^{-}_{\mathcal{B}^I_1}(x) = \mu^{-}_{\mathcal{B}^I_2}(x)$

(vi) $\vartheta^{+}_{\mathcal{B}^I_1}(x) = \vartheta^{+}_{\mathcal{B}^I_2}(x)$ and $\vartheta^{-}_{\mathcal{B}^I_1}(x) = \vartheta^{-}_{\mathcal{B}^I_2}(x)$

Definition 2.4. [11] Let $\tilde{\mathcal{C}}^I_1$ and $\tilde{\mathcal{C}}^I_2$ be two cubic bipolar intuitionistic fuzzy sets on $\mathcal{X}$. We say $\tilde{\mathcal{C}}^I_1 \subseteq_P \tilde{\mathcal{C}}^I_2$, if the followings are hold:

(i) $[\mu^{+l}_{\mathcal{B}^I_1}(x), \mu^{+u}_{\mathcal{B}^I_1}(x)] \leq [\mu^{+l}_{\mathcal{B}^I_2}(x), \mu^{+u}_{\mathcal{B}^I_2}(x)]$ (ii) $[\mu^{-l}_{\mathcal{B}^I_1}(x), \mu^{-u}_{\mathcal{B}^I_1}(x)] \geq [\mu^{-l}_{\mathcal{B}^I_2}(x), \mu^{-u}_{\mathcal{B}^I_2}(x)]$

(iii) $[\vartheta^{+l}_{\mathcal{B}^I_1}(x), \vartheta^{+u}_{\mathcal{B}^I_1}(x)] \geq [\vartheta^{+l}_{\mathcal{B}^I_2}(x), \vartheta^{+u}_{\mathcal{B}^I_2}(x)]$ (iv) $[\vartheta^{-l}_{\mathcal{B}^I_1}(x), \vartheta^{-u}_{\mathcal{B}^I_1}(x)] \leq [\vartheta^{-l}_{\mathcal{B}^I_2}(x), \vartheta^{-u}_{\mathcal{B}^I_2}(x)]$

(v) $\mu^{+}_{\mathcal{B}^{I_1}}(x) \leq \mu^{+}_{\mathcal{B}^{I_2}}(x)$ and $\mu^{-}_{\mathcal{B}^{I_1}}(x) \geq \mu^{-}_{\mathcal{B}^{I_2}}(x)$

(vi) $\vartheta^{+}_{\mathcal{B}^{I_1}}(x) \geq \vartheta^{+}_{\mathcal{B}^{I_2}}(x)$ and $\vartheta^{-}_{\mathcal{B}^{I_1}}(x) \leq \vartheta^{-}_{\mathcal{B}^{I_2}}(x)$

Definition 2.5. [12] Let $\tilde{\mathcal{C}}^{I}_1$ and $\tilde{\mathcal{C}}^{I}_2$ be two cubic bipolar intuitionistic fuzzy sets on $\mathcal{X}$. We say $\tilde{\mathcal{C}}^{I}_1 \subseteq_R \tilde{\mathcal{C}}^{I}_2$, if the followings are hold:

(i) $\left[\mu^{+l}_{\mathcal{B}^{I_1}}(x), \mu^{+u}_{\mathcal{B}^{I_1}}(x)\right] \leq \left[\mu^{+l}_{\mathcal{B}^{I_2}}(x), \mu^{+u}_{\mathcal{B}^{I_2}}(x)\right]$ (ii) $\left[\mu^{-l}_{\mathcal{B}^{I_1}}(x), \mu^{-u}_{\mathcal{B}^{I_1}}(x)\right] \geq \left[\mu^{-l}_{\mathcal{B}^{I_2}}(x), \mu^{-u}_{\mathcal{B}^{I_2}}(x)\right]$

(iii) $\left[\vartheta^{+l}_{\mathcal{B}^{I_1}}(x), \vartheta^{+u}_{\mathcal{B}^{I_1}}(x)\right] \geq \left[\vartheta^{+l}_{\mathcal{B}^{I_2}}(x), \vartheta^{+u}_{\mathcal{B}^{I_2}}(x)\right]$ (iv) $\left[\vartheta^{-l}_{\mathcal{B}^{I_1}}(x), \vartheta^{-u}_{\mathcal{B}^{I_1}}(x)\right] \leq \left[\vartheta^{-l}_{\mathcal{B}^{I_2}}(x), \vartheta^{-u}_{\mathcal{B}^{I_2}}(x)\right]$

(v) $\mu^{+}_{\mathcal{B}^{I_1}}(x) \geq \mu^{+}_{\mathcal{B}^{I_2}}(x)$ and $\mu^{-}_{\mathcal{B}^{I_1}}(x) \leq \mu^{-}_{\mathcal{B}^{I_2}}(x)$

(vi) $\vartheta^{+}_{\mathcal{B}^{I_1}}(x) \leq \vartheta^{+}_{\mathcal{B}^{I_2}}(x)$ and $\vartheta^{-}_{\mathcal{B}^{I_1}}(x) \geq \vartheta^{-}_{\mathcal{B}^{I_2}}(x)$

Definition 2.6. [12] The complement of CBIFS $\tilde{\mathcal{C}}^{I^c}$ on $\mathcal{X}$ can be defined as

$$\tilde{\mathcal{C}}^{I^c} = \{(x, (\left[\vartheta^{+l}_{\mathcal{B}^I}(x), \vartheta^{+u}_{\mathcal{B}^I}(x)\right], \left[\vartheta^{-l}_{\mathcal{B}^I}(x), \vartheta^{-u}_{\mathcal{B}^I}(x)\right], [\mu^{+l}_{\mathcal{B}^I}(x), \mu^{+u}_{\mathcal{B}^I}(x)], [\mu^{-l}_{\mathcal{B}^I}(x), \mu^{-u}_{\mathcal{B}^I}(x)]),$$
$$(\vartheta^{+}_{\mathcal{B}^I}(x), \vartheta^{-}_{\mathcal{B}^I}(x), \mu^{+}_{\mathcal{B}^I}(x), \mu^{-}_{\mathcal{B}^I}(x)) : x \in \mathcal{X}\}.$$

2.2 Some findings on CBIFSs

We will now go over a few CBIFS findings that will be used to define the CBIFS topology.

Definition 2.7. A CBIFS is referred to as empty CBIFS under P-order (denoted as φ_{C_P}) if $[\mu^{+l}_{\mathcal{B}^I}, \mu^{+u}_{\mathcal{B}^I}] = [0,0], [\mu^{-l}_{\mathcal{B}^I}, \mu^{-u}_{\mathcal{B}^I}] = [0,0], [\vartheta^{+l}_{\mathcal{B}^I}, \vartheta^{+u}_{\mathcal{B}^I}] = [1,1], [\vartheta^{-l}_{\mathcal{B}^I}, \vartheta^{-u}_{\mathcal{B}^I}] = [-1,-1]$ and $\left(\mu^{+}_{\mathcal{B}^I}, \mu^{-}_{\mathcal{B}^I}, \vartheta^{+}_{\mathcal{B}^I}, \vartheta^{-}_{\mathcal{B}^I}\right) = (0,0,1,-1)$. It can be expressed as $\varphi_{C_P} = \{([0,0], [0,0], [1,1], [-1,-1]),$

$(0,0,1,-1)\}.$

Definition 2.7. A CBIFS is referred to as empty CBIFS under R-order (denoted as φ_{C_R}) if $[\mu^{+l}_{\mathcal{B}^I}, \mu^{+u}_{\mathcal{B}^I}] = [0,0], [\mu^{-l}_{\mathcal{B}^I}, \mu^{-u}_{\mathcal{B}^I}] = [0,0], [\vartheta^{+l}_{\mathcal{B}^I}, \vartheta^{+u}_{\mathcal{B}^I}] = [1,1], [\vartheta^{-l}_{\mathcal{B}^I}, \vartheta^{-u}_{\mathcal{B}^I}] = [-1,-1]$ and $\left(\mu^{+}_{\mathcal{B}^I}, \mu^{-}_{\mathcal{B}^I}, \vartheta^{+}_{\mathcal{B}^I}, \vartheta^{-}_{\mathcal{B}^I}\right) = (1,-1,0,0)$. It can be expressed as $\varphi_{C_R} = \{([0,0], [0,0], [1,1], [-1,-1]),$

$(1,-1,0,0)\}.$

Definition 2.7. A CBIFS is referred to as whole CBIFS under P-order (denoted as Ω_{C_P}) if $[\mu^{+l}_{\mathcal{B}^I}, \mu^{+u}_{\mathcal{B}^I}] = [1,1], [\mu^{-l}_{\mathcal{B}^I}, \mu^{-u}_{\mathcal{B}^I}] = [-1,-1], [\vartheta^{+l}_{\mathcal{B}^I}, \vartheta^{+u}_{\mathcal{B}^I}] = [0,0], [\vartheta^{-l}_{\mathcal{B}^I}, \vartheta^{-u}_{\mathcal{B}^I}] = [0,0]$ and $\left(\mu^{+}_{\mathcal{B}^I}, \mu^{-}_{\mathcal{B}^I}, \vartheta^{+}_{\mathcal{B}^I}, \vartheta^{-}_{\mathcal{B}^I}\right) = (1,-1,0,0)$. It can be expressed as $\Omega_{C_P} = \{([1,1], [-1,-1], [0,0], [0,0]),$

$(1,-1,0,0)\}.$

Definition 2.7. A CBIFS is referred to as empty CBIFS under R-order (denoted as Ω_{C_R}) if $[\mu^{+l}_{\mathcal{B}^I}, \mu^{+u}_{\mathcal{B}^I}] = [1,1], [\mu^{-l}_{\mathcal{B}^I}, \mu^{-u}_{\mathcal{B}^I}] = [-1,-1], [\vartheta^{+l}_{\mathcal{B}^I}, \vartheta^{+u}_{\mathcal{B}^I}] = [0,0], [\vartheta^{-l}_{\mathcal{B}^I}, \vartheta^{-u}_{\mathcal{B}^I}] = [0,0]$ and $\left(\mu^{+}_{\mathcal{B}^I}, \mu^{-}_{\mathcal{B}^I}, \vartheta^{+}_{\mathcal{B}^I}, \vartheta^{-}_{\mathcal{B}^I}\right) = (1,-1,0,0)$. It can be expressed as $\Omega_{C_R} = \{([1,1], [-1,-1], [0,0], [0,0]),$

$(0,0,1,-1)\}.$

3 P-ORDER CUBIC BIPOLAR INTUITIONISTIC FUZZY

TOPOLOGY

Definition 3.1. Let $\mathcal{X}$ be a non-empty universal set and the collection of all CBIFS in $\mathcal{X}$ can be denoted as $\tilde{\mathcal{C}}^{I}(\mathcal{X})$. Now, consider τ_{C_P}, a collection of CBIFSs in $\mathcal{X}$ and it is said to form a cubic bipolar intuitionistic fuzzy topology under P-order if it satisfies the following properties:

1. φ_{C_P} and Ω_{C_P} belongs to τ_{C_P}.
2. $\cup_P(C_{P_j}) \in \tau_{C_P}$, where $C_{P_j} \in \tau_{C_P}$and $j \in J, J$ be an indexed set.
3. $\cap_P(C_{P_j}) \in \tau_{C_P}$, where $C_{P_j} \in \tau_{C_P}, j = 1,2, \ldots, n$.

Then τ_{C_P} is called CBIFS topology under P-order and the pair $(\mathcal{X}, \tau_{C_P})$ is denoted as PCBIFS topological space.

Example 3.2. Let $\mathcal{X} = \{x_1, x_{2,} x_3\}$ be the set and $\tilde{\mathcal{C}}^{I}(\mathcal{X})$ to be the collection of all CBIFS in $\mathcal{X}$. Let us consider two cubic bipolar intuitionistic fuzzy subsets of $\tilde{\mathcal{C}}^{I}(\mathcal{X})$ as:

$C_{P_1} = \{(x_1, ([0.31, 0.41], [-0.56, -0.49], [0.27, 0.37], [-0.33, -0.23]), (0.35, -0.53, 0.29, -0.25)), (x_2, ([0.55, 0.65], [-0.39, -0.29], [0.10, 0.20], [-0.40, -0.30]),$
(0.60, -0.35, 0.15, -0.35)), $(x_3, ([0.63, 0.72], [-0.50, -0.40], [0.10, 0.18], [-0.15,$
$-0.09]), (0.65, -0.45, 0.15, -0.11))\}$

$C_{P_2} = \{(x_1, ([0.26, 0.36], [-0.51, -0.43], [0.46, 0.58], [-0.34, -0.27]), (0.30, -0.46, 0.50, -0.28)), (x_2, ([0.51, 0.55], [-0.26, -0.20], [0.23, 0.33], [-0.56, -0.48]),$
(0.53, -0.23, 0.31, -0.50)), $(x_3, ([0.45, 0.55], [-0.29, -0.26], [0.20, 0.30], [-0.28,$
$-0.20]), (0.50, -0.22, 0.25, -0.23))\}$

Then, P-union and P-intersection of PCBIFSs C_{P_1} and C_{P_2} are demonstrated in the following Table 1 and Table 2:

Table 1: Union under P-order

$\cup_P$	φ_{C_P}	Ω_{C_P}	C_{P_1}	C_{P_2}
φ_{C_P}	φ_{C_P}	Ω_{C_P}	C_{P_1}	C_{P_2}
Ω_{C_P}	Ω_{C_P}	Ω_{C_P}	Ω_{C_P}	Ω_{C_P}
C_{P_1}	C_{P_1}	Ω_{C_P}	C_{P_1}	C_{P_1}
C_{P_2}	C_{P_2}	Ω_{C_P}	C_{P_1}	C_{P_2}

Table 2: Intersection under P-order

$\cap_P$	φ_{C_P}	Ω_{C_P}	C_{P_1}	C_{P_2}
φ_{C_P}	φ_{C_P}	φ_{C_P}	φ_{C_P}	φ_{C_P}
Ω_{C_P}	φ_{C_P}	Ω_{C_P}	C_{P_1}	C_{P_2}
C_{P_1}	φ_{C_P}	C_{P_1}	C_{P_1}	C_{P_2}
C_{P_2}	φ_{C_P}	C_{P_2}	C_{P_2}	C_{P_2}

We have, $\tau_{C_P} = \{\varphi_{C_P}, \Omega_{C_P}, C_{P_1}, C_{P_2}\}$

$\tau_{C_P}{}' = \{\varphi_{C_P}, \Omega_{C_P}, C_{P_1}\}$

$\tau_{C_P}{}'' = \{\varphi_{C_P}, \Omega_{C_P}, C_{P_2}\}$

$\tau_{C_P}{}''' = \{\varphi_{C_P}, \Omega_{C_P}\}$

Clearly, $\tau_{C_P}, \tau_{C_P}{}', \tau_{C_P}{}'', \tau_{C_P}{}'''$ are the cubic bipolar intuitionistic fuzzy topologies of $\mathcal{X}$ under P-order.

Definition 3.3. Let $\mathcal{X}$ be a non-empty set and $\tau_{C_P}^D = C_P(\mathcal{X})$ be represented as the collection of all CBIFSs in $\mathcal{X}$ with P-order. Then $\tau_{C_P}^D$ is said to be the discrete PCBIF topology on $\mathcal{X}$ and is considered to be the largest PCBIFS topology on $\mathcal{X}$.

Definition 3.4. Let $\mathcal{X}$ be a non-empty set and $\tau_{C_P}^I = \{\varphi_{C_P}, \Omega_{C_P}\}$ be represented the collection of empty and whole CBIFSs in $\mathcal{X}$ with P-order. Then $\tau_{C_P}^I$ is said to be the indiscrete PCBIF topology on $\mathcal{X}$ and is considered to be the smallest PCBIFS topology on $\mathcal{X}$.

Definition 3.5. The members of PCBIFS topology τ_{C_P} are referred to as P-cubic bipolar intuitionistic fuzzy open sets (PCBIFOS) in $(\mathcal{X}, \tau_{C_P})$.

Theorem 3.6. Consider $(\mathcal{X}, \tau_{C_P})$ be any PCBIFS topological space. Then,

1. φ_{C_P} and Ω_{C_P} are PCBIFOS.
2. For any arbitrary C_{P_j}, where each C_{P_j} is a PCBIFOS, $\cup_P(C_{P_j})$ is a PCBIFOS.
3. For any finite C_{P_j}, where each C_{P_j} is a PCBIFOS, $\cap_P(C_{P_j})$ is a PCBIFOS.

Proof.

1. According to the Definition 3.1., φ_{C_P} and $\Omega_{C_P} \in \tau_{C_P}$. Then, φ_{C_P} and Ω_{C_P} are obviously PCBIFOS.

2. Let $\{C_{P_j} | j \in J\}$ represents PCBIFOS. Every C_{P_j} is a member of τ_{C_P}. In accordance with the Definition 3.1., $\cup_P(C_{P_j})$ is evidently PCBIFOS.

3. Let $\{C_{P_j} | j = 1,2, \dots, n\}$ represents PCBIFOS. Every C_{P_j} is a member of τ_{C_P}. In accordance with the Definition 3.1., $\cap_P(C_{P_j})$ is evidently PCBIFOS.

Definition 3.7. The complement of P-cubic bipolar intuitionistic fuzzy open sets in $(\mathcal{X}, \tau_{C_P})$ are said to be the P-cubic bipolar intuitionistic fuzzy closed sets (PCBIFCS) in $(\mathcal{X}, \tau_{C_P})$.

Theorem 3.8. Consider $(\mathcal{X}, \tau_{C_P})$ be any PCBIFS topological space. Then,

1. φ_{C_P} and Ω_{C_P} are PCBIFCS.

2. For any arbitrary C_{P_j}, where each C_{P_j} is a PCBIFCS, $\cap_P(C_{P_j})$ is a PCBIFCS.

3. For any finite C_{P_j}, where each C_{P_j} is a PCBIFCS, $\cup_P(C_{P_j})$ is a PCBIFCS.

Proof. The proof is similar to Theorem 3.6.

Definition 3.9. The PCBIFSs which are both PCBIFOSs and PCBIFCSs are referred as PCBIF clopen sets in $(\mathcal{X}, \tau_{C_P})$.

Remark 3.10.
1. For every PCBIF topological space $(\mathcal{X}, \tau_{C_P})$, we have both φ_{C_P} and Ω_{C_P} are PCBIF Clopen Sets.

2. Each CBIFS in $\mathcal{X}$ is a PCBIF clopen sets for every discrete PCBIFS topology.

3. In indiscrete PCBIFS topology, φ_{C_P} and Ω_{C_P} are the only PCBIF Clopen Sets.

Definition 3.11. Consider $\tau_{C_{P_1}}$ and $\tau_{C_{P_2}}$ be the two PCBIFS topologies in $\mathcal{X}$. If $\tau_{C_{P_1}} \subseteq_P \tau_{C_{P_2}}$, then they are said to be comparable. Also, $\tau_{C_{P_1}}$ is PCBIF coarser than $\tau_{C_{P_2}}$ and $\tau_{C_{P_2}}$ is PCBIF finer than $\tau_{C_{P_1}}$.

Example 3.12. Consider Example 3.2, take $\tau_{C_P} = \{\varphi_{C_P}, \Omega_{C_P}, C_{P_1}, C_{P_2}\}$ and $\tau_{C_P}' = \{\varphi_{C_P}, \Omega_{C_P}, C_{P_1}\}$ be the two PCBIF topologies on $\mathcal{X}$. Then $\tau_{C_P}' \subseteq_P \tau_{C_P}$. Hence,$\tau_{C_P}'$ is PCBIF coarser than τ_{C_P}.

Definition 3.13. Consider $(\mathcal{X}, \tau_{C_{P_X}})$ be the PCBIFS topology. Let Y ⊆ X and $\tau_{C_{P_Y}}$be the PCBIFS topology on Y whose PCBIFOSs are $C_{P_Y} = C_{P_X} \cap_P Y$, where C_{P_X} are PCBIFOSs of $\tau_{C_{P_X}}$ and C_{P_Y} are PCBIFOSs of $\tau_{C_{P_Y}}$. Then $\tau_{C_{P_Y}}$ is a PCBIF subspace of $\tau_{C_{P_X}}$ (i.e.) $\tau_{C_{P_Y}} = \{C_{P_Y}: C_{P_Y} = C_{P_X} \cap_P Y, C_{P_X} \in \tau_{C_{P_X}}\}$.

Definition 3.14. Consider $(\mathcal{X}, \tau_{C_P})$ be the PCBIFS topology and let $C_P \in \tilde{C}^I(\mathcal{X})$, then the interior of C_P be denoted as $int\ (C_P)$ and defined to be the union of all open CBIF subsets contained in C_P. It is considered to be the greatest open PCBIFS contained in C_P.

We have $int\ (C_P) = \{\cup_P \left(C_{P_j}\right) | C_{P_j} \subseteq C_P \ and\ C_P \in \tau_{C_{P_j}}, j \in J\}$

Example 3.15. Consider PCBIF topologies as constructed in Example 3.2 and let $C_{P_3} \in \tilde{C}^I(\mathcal{X})$ given as,

$C_{P_3} = \{(x_1, ([0.45, 0.56], [-0.70, -0.60], [0.10, 0.20], [-0.20, -0.18]), (0.50, -0.65, 0.15, -0.19)), (x_2, ([0.70, 0.74], [-0.40, -0.33], [0.11, 0.23], [-0.23, -0.16]),$

(0.73, -0.35, 0.20, -0.20)), $(x_3, ([0.80, 0.89], [-0.61, -0.48], [0.05, 0.10], [-0.10,$

$-0.08]), (0.85, -0.53, 0.07, -0.09))\}$

Then, $int\left(C_{P_3}\right) = \cup_P\{\varphi_{C_P}, C_{P_1}, C_{P_2}\} = C_{P_1}$

Theorem 3.16. Let τ_{C_P} be the PCBIFS topology on $\mathcal{X}$ and consider $C_P \in \tilde{C}^I(\mathcal{X})$. Then C_P is open if and only if $int\ (C_P) = C_P$.

Proof. If C_P is open, by Definition 3.14, interior of C_P is the union of all open CBIF sets contained in C_P. Since C_P is open and contained in C_P, we have $C_P \subseteq_P int\ (C_P)$. Also, we have $int(C_P) \subseteq_P C_P$.

$$\Rightarrow int(C_P) = C_P.$$

Conversely, assume that $int(C_P) = C_P$. By definition 3.14, the interior of C_P is the union of open sets, so it is open.

$\Rightarrow C_P$ is open.

Therefore, C_P is open if and only if $int(C_P) = C_P$.

Theorem 3.17. Consider $(\mathcal{X}, \tau_{C_P})$ be the PCBIFS topology and let $C_P, C_{P_1}, C_{P_2} \in \tilde{C}^I(\mathcal{X})$.

Then we have,

1. $int(int(C_P)) = int(C_P)$

2. If $C_{P_1} \subseteq_P C_{P_2}$, then $int\left(C_{P_1}\right) \subseteq_P int(C_{P_2})$

3. $int(C_{P_1} \cap_P C_{P_2}) = int(C_{P_1}) \cap_P int(C_{P_2})$

4. $int(C_{P_1}) \cup_P int(C_{P_2}) \subseteq_P int(C_{P_1} \cup_P C_{P_2})$

Proof:

1. Since by Definition 3.14, $int(C_P) \subseteq_P C_P$. By applying monotonicity of the interior operator in CBIF setting gives,

$int(int(C_P)) \subseteq_P int\ (C_P)$ (2)

Also, $int\ (C_P)$ is itself a CBIF open set, every open set is contained in the interior itself.

Hence,

$$int(C_P) \subseteq_P int(int((C_P)) \quad (3)$$

From (2) and (3), we get $int(int(C_P)) = int\ (C_P)$

2. By Defintion 3.14, $int\ (C_P)$ is a CBIF open set and $int(C_P) \subseteq_P C_P$. Given $C_{P_1} \subseteq_P C_{P_2}$, we have $int(C_{P_1}) \subseteq_P C_{P_2}$

$\Rightarrow int\ (int(C_{P_1})) \subseteq_P int(C_{P_2})$

$\Rightarrow int(C_{P_1}) \subseteq_P int(C_{P_2})$

3. From Definition 3.14, $int(C_{P_1}) \subseteq_P C_{P_1}$and $int(C_{P_2}) \subseteq_P C_{P_2}$.

By Monotonicity, we have $int(C_{P_1}) \cap_P int(C_{P_2}) \subseteq_P C_{P_1} \cap_P C_{P_2}$.

Since $int(C_{P_1} \cap_P C_{P_2})$ is the largest CBIF open set contained in $C_{P_1} \cap_P C_{P_2}$.

We must have

$int(C_{P_1}) \cap_P int(C_{P_2}) \subseteq_P int(C_{P_1} \cap_P C_{P_2})$. (4)

Also, we know that $C_{P_1} \cap_P C_{P_2} \subseteq_P C_{P_1}$ and $C_{P_1} \cap_P C_{P_2} \subseteq_P C_{P_2}$

$\Rightarrow int(C_{P_1} \cap_P C_{P_2}) \subseteq_P int(C_{P_1})$ and $int(C_{P_1} \cap_P C_{P_2}) \subseteq_P int(C_{P_2})$.

$int(C_{P_1} \cap_P C_{P_2}) \subseteq_P int(C_{P_1}) \cap_P int(C_{P_2})$ (5)

From (4) and (5), we have $int(C_{P_1} \cap_P C_{P_2}) = int(C_{P_1}) \cap_P int(C_{P_2})$.

4. From Definition 3.14, $int(C_{P_1}) \subseteq_P C_{P_1}$and $int(C_{P_2}) \subseteq_P C_{P_2}$.

By Monotonicity, we have $int(C_{P_1}) \cup_P int(C_{P_2}) \subseteq_P C_{P_1} \cup_P C_{P_2}$.

Also, $int(C_{P_1} \cup_P C_{P_2})$ is the largest open set contained in $C_{P_1} \cup_P C_{P_2}$

$\Rightarrow int(C_{P_1}) \cup_P int(C_{P_2}) \subseteq_P int(C_{P_1} \cup_P C_{P_2})$

Definition 3.18. Consider $(\mathcal{X}, \tau_{C_P})$ be the PCBIFS topology and let $C_P \in \tilde{C}^I(\mathcal{X})$, then the closure of C_P be denoted as $cl(C_P)$ and defined to be the intersection of all closed CBIF subsets contains C_P. It is referred to be the smallest closed PCBIF set containing C_P.

$$cl(C_P) = \{\cap_P (C_{P_j}) | C_P \subseteq_P C_{P_j}\ and\ (C_P)^c \in \tau_{C_{P_j}}, j \in J\}$$

Example 3.19. Consider CBIF topologies as constructed in Example 3.2, and the closed CBIFSs are $(\varphi_{C_P})^c = \Omega_{C_P}, (\Omega_{C_P})^c = \varphi_{C_P}$,

$(C_{P_1})^c = \{(x_1, ([0.27, 0.37], [-0.33, -0.23], [0.31, 0.41], [-0.56, -0.49]), (0.29, -0.25$
$0.35, -0.53)), (x_2, ([0.10, 0.20], [-0.40, -0.30], [0.55, 0.65], [-0.39, -0.29]),$

$(0.15, -0.35, 0.60, -0.35)), (x_3, ([0.10, 0.18], [-0.15, -0.09], [0.63, 0.72], [-0.50,$

$-0.40]), (0.15, -0.11, 0.65, -0.45))\}$

$(C_{P_2})^c = \{(x_1, ([0.46, 0.58], [-0.34, -0.27], [0.26, 0.36], [-0.51, -0.43]), (0.50, -0.28,$
$0.30, -0.46)), (x_2, ([0.23, 0.33], [-0.56, -0.48], [0.51, 0.55], [-0.26, -0.20]),$

$(0.31, -0.50, 0.53, -0.23)), (x_3, ([0.20, 0.30], [-0.28, -0.20], [0.45, 0.55], [-0.29,$

$-0.26]), (0.25, -0.23, 0.50, -0.22))\}$

Now, consider $C_{P_4} \in \tilde{\mathcal{C}}^I(\mathcal{X})$ given as,

$C_{P_4} = \{(x_1, ([0.05, 0.10], [-0.20, -0.15], [0.40, 0.50], [-0.60, -0.51]), (0.07, -0.17, 0.43, -0.55)), (x_2, ([0.07, 0.16], [-0.34, -0.28], [0.60, 0.70], [-0.43, -0.33]),$

$(0.10, -0.30, 0.65, -0.40)), (x_3, ([0.09, 0.12], [-0.07, -0.08], [0.70, 0.77], [-0.60,$

$-0.50]), (0.11, -0.08, 0.71, -0.55))\}$

Then, $cl(C_{P_4}) = \cap_P \{\Omega_{C_P}, (C_{P_1})^c, (C_{P_2})^c\} = (C_{P_2})^c$

Theorem 3.20. Let τ_{C_P} be the PCBIFS topology on $\mathcal{X}$ and consider $C_P \in \tilde{\mathcal{C}}^I(\mathcal{X})$. Then C_P is closed if and only if $cl(C_P) = C_P$.

Proof. If C_P is closed, by Definition 3.18, closure of C_P is the intersection of all closed CBIF sets containing C_P. Since C_P is closed and contains C_P, we have

$cl(C_P) \subseteq_P C_P$ (6)

But always

$(C_P) \subseteq_P cl(C_P)$ (7)

From (6) and (7), we get $cl(C_P) = C_P$.

Conversely, assume that $cl(C_P) = C_P$. We know that the closure of C_P is always a closed set since it is the infimum of CBIF closed sets.

$\Longrightarrow C_P$ must be a closed set.

Therefore, C_P is closed if and only if $cl(C_P) = C_P$.

Theorem 3.21. Let τ_{C_P} be the PCBIFS topology on $\mathcal{X}$ and consider $C_P, C_{P_1}, C_{P_2} \in \tilde{\mathcal{C}}^I(\mathcal{X})$. Then we have,

1. $cl(cl(C_P)) = cl(C_P)$
2. If $C_{P_1} \subseteq_P C_{P_2}$, then $cl(C_{P_1}) \subseteq_P cl(C_{P_2})$
3. $cl(C_{P_1} \cup_P C_{P_2}) = cl(C_{P_1}) \cup_P cl(C_{P_2})$
4. $cl(C_{P_1} \cap_P C_{P_2}) \subseteq_P cl(C_{P_1}) \cap_P cl(C_{P_2})$

Proof.
1. We know that, for any $C_P, C_P \subseteq_P cl(C_P)$. So we have,

$cl(C_P) \subseteq_P cl(cl((C_P))$ (8)

By the Definition 3.18, the set $cl(C_P)$ is closed. Hence, $cl(C_P)$ is one of the closed sets

containing $cl(C_P)$.

Therefore, intersection of all closed sets containing $cl(C_P)$ contained in $cl(C_P)$.

$cl(cl((C_P)) \subseteq_P cl(C_P)$.

From (8) and (9), (9)

$$cl(cl(C_P)) = cl(C_P)$$

2. From the Definition 3.18, $cl(C_{P_2})$ is the smallest closed set containing C_{P_2}.

That is, $C_{P_2} \subseteq_P cl(C_{P_2})$

Since $C_{P_1} \subseteq_P C_{P_2}$, from above we have $C_{P_1} \subseteq_P cl(C_{P_2})$.

But $cl(C_{P_1})$ is the smallest closed set containing C_{P_1}, we have $cl(C_{P_1}) \subseteq_P cl(C_{P_2})$.

3. The sets $cl(C_{P_1})$ and $cl(C_{P_2})$ are closed, since the finite union of closed sets is closed, we have $cl(C_{P_1}) \cup_P cl(C_{P_2})$ is closed.

Also, $C_{P_1} \subseteq_P cl(C_{P_1}) \subseteq_P cl(C_{P_1}) \cup_P cl(C_{P_2})$ and $C_{P_2} \subseteq_P cl(C_{P_2}) \subseteq_P cl(C_{P_1}) \cup_P cl(C_{P_2})$.

$\Longrightarrow C_{P_1} \cup_P C_{P_2} \subseteq_P cl(C_{P_1}) \cup_P cl(C_{P_2})$.

Since, $cl(C_{P_1}) \cup_P cl(C_{P_2})$ is the smallest CBIF closed set containing $cl(C_{P_1}) \cup_P cl(C_{P_2})$.

Therefore, we get

$cl(C_{P_1} \cup_P C_{P_2}) \subseteq_P cl(C_{P_1}) \cup_P cl(C_{P_2})$ (10)

Now, we know $C_{P_1} \subseteq_P cl(C_{P_1}) \cup_P cl(C_{P_2})$ and $C_{P_2} \subseteq_P cl(C_{P_1}) \cup_P cl(C_{P_2})$

$\Longrightarrow cl(C_{P_1}) \subseteq_P cl(C_{P_1}) \cup_P cl(C_{P_2})$ and $cl(C_{P_2}) \subseteq_P cl(C_{P_1}) \cup_P cl(C_{P_2})$

$\Longrightarrow cl(C_{P_1}) \cup_P cl(C_{P_2}) \subseteq_P cl(C_{P_1} \cup_P C_{P_2})$ (11)

From (10) and (11), $cl(C_{P_1}) \cup_P cl(C_{P_2}) = cl(C_{P_1} \cup_P C_{P_2})$.

4. We know that $cl(C_P)$ is the smallest closed set containing C_P. Also, $cl(C_{P_1})$ is closed and contains C_{P_1} and $cl(C_{P_2})$ is closed and contains C_{P_2}.

Since intersection of two closed CBIFS is closed, we have $cl(C_{P_1}) \cap_P cl(C_{P_2})$ is also closed and contains $C_{P_1} \cap_P C_{P_2}$.

$\Longrightarrow C_{P_1} \cap_P C_{P_2} \subseteq_P cl(C_{P_1}) \cap_P cl(C_{P_2})$.

Also, we know that $cl(C_{P_1} \cap_P C_{P_2})$ is the smallest closed set containing $C_{P_1} \cap_P C_{P_2}$.

Therefore, we get $cl(C_{P_1} \cap_P C_{P_2}) \subseteq_P cl(C_{P_1}) \cap_P cl(C_{P_2})$.

Definition 3.22. Consider C_P be the PCBIF subset of $(\mathcal{X}, \tau_{C_P})$, then the Frontier of C_P to be defined as, $Fr(C_P) = cl(C_P) \cap_P cl((C_P)^c)$.

Definition 3.23. Consider CP be the PCBIF subset of $(\mathcal{X}, \tau_{C_P})$, then the Exterior of C_P to be defined as, $Ext(C_P) = int((C_P)^c) = (cl(C_P))^c$.

4 R-ORDER CUBIC BIPOLAR INTUITIONISTIC FUZZY TOPOLOGY

Definition 4.1. Let $\mathcal{X}$ be a non-empty universal set and the collection of all CBIFS in $\mathcal{X}$ can be denoted as $\tilde{C}^I(\mathcal{X})$. Now, consider τ_{C_P}, a collection of CBIFSs in $\mathcal{X}$ and it is said to form a cubic bipolar intuitionistic fuzzy topology under R-order if it satisfies the following properties:

1. φ_{C_R} and Ω_{C_R} belongs to τ_{C_R}.
2. $\cup_R(C_{R_j}) \in \tau_{C_R}$, where $C_{R_j} \in \tau_{C_R}$ and $j \in J, J$ be an indexed set.
3. $\cap_R(C_{R_j}) \in \tau_{C_R}$, where $C_{R_j} \in \tau_{C_R}, j = 1,2, \dots, n$.

Then τ_{C_R} is called CBIFS topology under R-order and the pair $(\mathcal{X}, \tau_{C_R})$ is denoted as RCBIFS topological space.

Example 4.2. Let $\mathcal{X} = \{x_1, x_{2,} x_3\}$ be the set and $\tilde{C}^I(\mathcal{X})$ to be the collection of all CBIFS in $\mathcal{X}$. Let us consider two cubic bipolar intuitionistic fuzzy subsets of $\tilde{C}^I(\mathcal{X})$ as:

$C_{R_1} = \{(x_1, ([0.15, 0.20], [-0.36, -0.29], [0.40, 0.49], [-0.25, -0.20]), (0.50, -0.48, 0.33, -0.15)), (x_2, ([0.23, 0.30], [-0.39, -0.34], [0.50, 0.56], [-0.43, -0.37]),$

$(0.60, -0.50, 0.10, -0.11)), (x_3, ([0.21, 0.31], [-0.27, -0.18], [0.60, 0.67], [-0.07,$

$-0.03]), (0.19, -0.60, 0.50, -0.02))\}$

$C_{R_2} = \{(x_1, ([0.36, 0.40], [-0.50, -0.45], [0.27, 0.36], [-0.15, -0.11]), (0.45, -0.40, 0.42, -0.20)), (x_2, ([0.50, 0.57], [-0.26, -0.18], [0.10, 0.16], [-0.46, -0.39]),$

$(0.38, -0.30, 0.40, -0.50)), (x_3, ([0.25, 0.35], [-0.43, -0.33], [0.48, 0.54], [-0.10,$

$-0.08]), (0.16, -0.48, 0.67, -0.06))\}$

Then, R-union and R-intersection of RCBIFSs C_{R_1} and C_{R_2} are demonstrated in the following Table 3 and Table 4:

Table 3: Union under R-order

$\cup_R$	φ_{C_R}	Ω_{C_R}	C_{R_1}	C_{R_2}
φ_{C_R}	φ_{C_R}	Ω_{C_R}	C_{R_1}	C_{R_2}
Ω_{C_R}	Ω_{C_R}	Ω_{C_R}	Ω_{C_R}	Ω_{C_R}
C_{R_1}	C_{R_1}	Ω_{C_R}	C_{R_1}	C_{R_2}
C_{R_2}	C_{R_2}	Ω_{C_R}	C_{R_2}	C_{R_2}

Table 4: Intersection under R-order

$\cap_R$	φ_{C_R}	Ω_{C_R}	C_{R_1}	C_{R_2}
φ_{C_R}	φ_{C_R}	φ_{C_R}	φ_{C_R}	φ_{C_R}
Ω_{C_R}	φ_{C_R}	Ω_{C_R}	C_{R_1}	C_{R_2}
C_{R_1}	φ_{C_R}	C_{R_1}	C_{R_1}	C_{R_1}
C_{R_2}	φ_{C_R}	C_{R_2}	C_{R_1}	C_{R_2}

We have, $\tau_{C_R} = \{\varphi_{C_R}, \Omega_{C_R}, C_{R_1}, C_{R_2}\}$

$\tau_{C_R}{}' = \{\varphi_{C_R}, \Omega_{C_R}, C_{R_1}\}$

$\tau_{C_R}{}'' = \{\varphi_{C_R}, \Omega_{C_R}, C_{R_2}\}$

$\tau_{C_R}{}''' = \{\varphi_{C_R}, \Omega_{C_R}\}$

Clearly, $\tau_{C_R}, \tau_{C_R}{}', \tau_{C_R}{}'', \tau_{C_R}{}'''$ are the cubic bipolar intuitionistic fuzzy topologies of $\mathcal{X}$ under R-order.

Definition 4.3. Let $\mathcal{X}$ be a non-empty set and $\tau_{C_R}^D = C_R(\mathcal{X})$ be represented as the collection of all CBIFSs in $\mathcal{X}$ with R-order. Then $\tau_{C_R}^D$ is said to be the discrete RCBIF topology on $\mathcal{X}$ and is considered to be the largest RCBIFS topology on $\mathcal{X}$.

Definition 4.4. Let $\mathcal{X}$ be a non-empty set and $\tau_{C_R}^I = \{\varphi_{C_R}, \Omega_{C_R}\}$ be represented the collection of empty and whole CBIFSs in $\mathcal{X}$ with R-order. Then $\tau_{C_R}^I$ is said to be the indiscrete RCBIF topology on $\mathcal{X}$ and is considered to be the smallest RCBIFS topology on $\mathcal{X}$.

Definition 4.5. The members of RCBIFS topology τ_{C_R} are referred to as R-cubic bipolar intuitionistic fuzzy open sets (RCBIFOS) in $(\mathcal{X}, \tau_{C_R})$.

Theorem 4.6. Consider $(\mathcal{X}, \tau_{C_R})$ be any RCBIFS topological space. Then,

1. φ_{C_R} and Ω_{C_R} are RCBIFOS.

2. For any arbitrary C_{R_j}, where each C_{R_j} is a RCBIFOS, $\cup_R(C_{R_j})$ is a RCBIFOS.

3. For any finite C_{R_j}, where each C_{R_j} is a RCBIFOS, $\cap_R(C_{R_j})$ is a RCBIFOS.

Proof.
1. According to the Definition 4.1., φ_{C_R} and $\Omega_{C_R} \in \tau_{C_R}$. Then, φ_{C_R} and Ω_{C_R} are obviously PCBIFOS.

2. Let $\{C_{R_j} | j \in J\}$ represents RCBIFOS. Every C_{R_j} is a member of τ_{C_R}. In accordance with the Definition 4.1., $\cup_R(C_{R_j})$ is evidently RCBIFOS.

3. Let $\{C_{R_j} | j = 1,2, \dots, n\}$ represents RCBIFOS. Every C_{R_j} is a member of τ_{C_R}. In accordance with the Definition 4.1., $\cap_R(C_{R_j})$ is evidently RCBIFOS.

Definition 4.7. The complement of R-cubic bipolar intuitionistic fuzzy open sets in $(\mathcal{X}, \tau_{C_R})$ are said to be the R-cubic bipolar intuitionistic fuzzy closed sets (RCBIFCS) in $(\mathcal{X}, \tau_{C_R})$.

Theorem 4.8. Consider $(\mathcal{X}, \tau_{C_R})$ be any RCBIFS topological space. Then,

1. φ_{C_R} and Ω_{C_R} are RCBIFCS.

2. For any arbitrary C_{R_j}, where each C_{R_j} is a RCBIFCS, $\cap_R(C_{R_j})$ is a RCBIFCS.

3. For any finite C_{R_j}, where each C_{R_j} is a RCBIFCS, $\cup_R(C_{R_j})$is a RCBIFCS.

Proof. The proof is similar to Theorem 4.6.

Definition 4.9. The RCBIFSs which are both RCBIFOSs and RCBIFCSs are referred as RCBIF clopen sets in $(\mathcal{X}, \tau_{C_R})$.

Remark 4.10.

1. For every RCBIF topological space $(\mathcal{X}, \tau_{C_R})$, we have both φ_{C_R} and Ω_{C_R} are RCBIF Clopen Sets.

2. Each CBIFS in $\mathcal{X}$ is a RCBIF clopen sets for every discrete RCBIFS topology.

3. In indiscrete RCBIFS topology, φ_{C_R} and Ω_{C_R} are the only RCBIF Clopen Sets.

Definition 4.11. Consider $\tau_{C_{R_1}}$ and $\tau_{C_{R_2}}$ be the two RCBIFS topologies in $\mathcal{X}$. If $\tau_{C_{R_1}} \subseteq_R \tau_{C_{R_2}}$, then they are said to be comparable. Also, $\tau_{C_{R_1}}$ is RCBIF coarser than $\tau_{C_{R_2}}$ and $\tau_{C_{R_2}}$ is RCBIF finer than $\tau_{C_{R_1}}$.

Example 4.12. Consider Example 4.2, take $\tau_{C_R} = \{\varphi_{C_R}, \Omega_{C_R}, C_{R_1}, C_{R_2}\}$ and $\tau_{C_R}{}' = \{\varphi_{C_R}, \Omega_{C_R}, C_{R_1}\}$ be the two RCBIF topologies on $\mathcal{X}$. Then $\tau_{C_R}{}' \subseteq_R \tau_{C_R}$. Hence,$\tau_{C_R}{}'$ is RCBIF coarser than τ_{C_R}.

Definition 4.13. Consider $(\mathcal{X}, \tau_{C_{R_X}})$ be the RCBIFS topology. Let Y ⊆ X and $\tau_{C_{R_Y}}$be the RCBIFS topology on Y whose RCBIFOSs are $C_{R_Y} = C_{R_X} \cap_R Y$, where C_{R_X} are RCBIFOSs of $\tau_{C_{R_X}}$ and C_{R_Y} are RCBIFOSs of $\tau_{C_{R_Y}}$. Then $\tau_{C_{R_Y}}$ is a RCBIF subspace of $\tau_{C_{R_X}}$ (i.e.) $\tau_{C_{R_Y}} = \{C_{R_Y} : C_{R_Y} = C_{R_X} \cap_R Y, C_{R_X} \in \tau_{C_{R_X}}\}$.

Definition 4.14. Consider $(\mathcal{X}, \tau_{C_R})$ be the RCBIFS topology and let $C_R \in \tilde{\mathcal{C}}^I(\mathcal{X})$, then the interior of C_R be denoted as $int\ (C_R)$ and defined to be the union of all open CBIF subsets contained in C_R. It is considered to be the greatest open RCBIFS contained in C_R.

We have $int\ (C_R) = \{\cup_R \left(C_{R_j}\right) | C_{R_j} \subseteq_R C_R\ and\ C_R \in \tau_{C_{R_j}}, j \in J\}$

Example 4.15. Consider RCBIF topologies as constructed in Example 4.2 and let $C_{R_3} \in \tilde{\mathcal{C}}^I(\mathcal{X})$ given as,

$C_{R_3} = \{(x_1, ([1,1], [-1,-1], [0,0], [0,0]), (0.35, -0.20, 0.60, -0.40)), (x_2, ([1,1], [-1,-1],$
$[0,0], [0,0,]), (0.10, -0.37, 0.79, -0.36)), x_3, ([0.43, 0.47], [-0.50, -0.43], [0.34, 0.41],$

$[-0.05, -0.02]), (0.12, -0.30, 0.70, -0.11))\}$

Then, $int\ (C_{R_3}) = \cup_R\{\varphi_{C_R}, C_{R_1}, C_{R_2}\} = C_{R_2}$

Theorem 4.16. Let τ_{C_R} be the RCBIFS topology on $\mathcal{X}$ and consider $C_R \in \tilde{\mathcal{C}}^I(\mathcal{X})$. Then C_R is open if and only if $int\ (C_R) = C_R$.

Proof. The proof is similar to Theorem 3.16.

Theorem 4.17. Consider $(\mathcal{X}, \tau_{C_R})$ be the RCBIFS topology and let $C_R, C_{R_1}, C_{R_2} \in \tilde{\mathcal{C}}^I(\mathcal{X})$.

Then we have,

1. $int(int(C_R)) = int(C_R)$

2. If $C_{R_1} \subseteq_R C_{R_2}$, then $int(C_{R_1}) \subseteq_R int(C_{R_2})$

3. $int(C_{R_1} \cap_R C_{R_2}) = int(C_{R_1}) \cap_R int(C_{R_2})$

4. $int(C_{R_1}) \cup_P int(C_{R_2}) \subseteq_R int(C_{R_1} \cup_R C_{R_2})$

Proof: The proof is similar to Theorem 3.17.

Definition 4.18. Consider $(\mathcal{X}, \tau_{C_R})$ be the RCBIFS topology and let $C_R \in \tilde{\mathcal{C}}^I(\mathcal{X})$, then the closure of C_R be denoted as $cl(C_R)$ and defined to be the intersection of all closed CBIF subsets contains C_R. It is referred to be the smallest closed RCBIF set containing C_R.

We have $cl(C_R) = \{\cap_R\ (C_{R_j}) | C_R \subseteq_R C_{R_j}\ and\ (C_R)^c \in \tau_{C_{R_j}}, j \in J\}$

Example 4.19. Consider CBIF topologies as constructed in Example 4.2, and the closed CBIFSs are $(\varphi_{C_R})^c = \Omega_{C_R}, (\Omega_{C_R})^c = \varphi_{C_R}$,

$(C_{R_1})^c = \{(x_1, ([0.40, 0.49], [-0.25, -0.20], [0.15, 0.20], [-0.36, -0.29]), (0.33, -0.15$
$0.50, -0.48)), (x_2, ([0.50, 0.56], [-0.43, -0.37], [0.23, 0.30], [-0.39, -0.34]),$

$(0.10, -0.11, 0.60, -0.50)), (x_3, ([0.60, 0.67], [-0.07, -0.03], [0.21, 0.31], [-0.27,$

$-0.18]), (0.50, -0.02, 0.19, -0.60))\}$

$(C_{R_2})^c = \{(x_1, ([0.27, 0.36], [-0.15, -0.11], [0.36, 0.40], [-0.50, -0.45]), (0.42, -0.20,$
$0.45, -0.40)), (x_2, ([0.10, 0.16], [-0.46, -0.39], [0.50, 0.57], [-0.26, -0.18]),$

$(0.40, -0.50, 0.38, -0.30)), (x_3, ([0.48, 0.54], [-0.10, -0.08], [0.25, 0.35], [-0.43,$

$-0.33]),(0.67,-0.06,0.16,-0.48))\}$

Now, consider $C_{R_4} \in \tilde{C}^I(\mathcal{X})$ given as,

$C_{R_4} = \{(x_1,([0.10,0.15],[-0.10,-0.07],[0.43,0.51],[-0.63,-0.57]),(0.50,-0.35,$
$0.36,-0.25)),(x_2,([0.03,0.12],[-0.20,-0.13],[0.64,0.70],[-0.41,-0.37]),$

$(0.46, -0.60, 0.20, -0.26)),(x_3,([0.50,0.53],[-0.33,-0.24],[0.20,0.26],[-0.50,$

$-0.43]),(0.67,-0.06,0.16,-0.48))\}$

Then, $cl(C_{R_4}) = \cap_R \{\Omega_{C_R},(C_{R_1})^c,(C_{R_2})^c\} = (C_{R_1})^c$

Theorem 4.20. Let τ_{C_R} be the RCBIFS topology on $\mathcal{X}$ and consider $C_R \in \tilde{C}^I(\mathcal{X})$. Then C_R is closed if and only if $cl(C_R) = C_R$.

Proof. The proof is similar to Theorem 3.20.

Theorem 4.21. Let τ_{C_R} be the RCBIFS topology on $\mathcal{X}$ and consider $C_R, C_{R_1}, C_{R_2} \in \tilde{C}^I(\mathcal{X})$. Then we have,

1. $cl(cl(C_R)) = cl(C_R)$
2. If $C_{R_1} \subseteq_R C_{R_2}$, then $cl(C_{R_1}) \subseteq_R cl(C_{R_2})$
3. $cl(C_{R_1} \cup_R C_{R_2}) = cl(C_{R_1}) \cup_R cl(C_{R_2})$
4. $cl(C_{R_1} \cap_R C_{R_2}) \subseteq_R cl(C_{R_1}) \cap_R cl(C_{R_2})$

Proof. The proof is similar to Theorem 3.21.

Definition 4.22. Consider C_R be the RCBIF subset of $(\mathcal{X},\tau_{C_R})$, then the Frontier of C_R to be

defined as, $Fr(C_R) = cl(C_R) \cap_R cl((C_R)^c)$.

Definition 4.23. Consider C_R be the RCBIF subset of $(\mathcal{X},\tau_{C_R})$, then the Exterior of C_R to be defined as, $Ext(C_R) = int((C_R)^c) = (cl(C_R))^c$.

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