Fuzzy Matrix Representations of Semi group Subsets and Their Inverse Structures

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Abstract
We present the concept of fuzzy-like semi group subsets within the framework of matrix representation and examine their inverse properties. This paper redefines fuzzy inverse subsets in semi groups through matrix- based formulations, analyzing their structural behavior under diverse algebraic operations. Additionally, we introduce a matrix interpretation of fuzzy-like semi group subsets, extending traditional semi group hypothesis into the fuzzy matrix domain. The research explores the interaction between matrix subsets and their inverse properties, contributing acumen into their algebraic importance. These findings contribute to advancing studies in fuzzy algebra, matrix theory and computational application.

Keywords – Fuzzy matrix, Fuzzy-like semi group, inverse properties, fuzzy algebra, semi group theory, algebraic structures.
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1. Introduction

The investigation of algebraic functions including fuzzy has been an important region of study, since the preamble of fuzzy set theory by [18]. A fuzzy subset of a set with elements X is stated as a mapping A: X → [0, 1], where [0, 1] characterize the degree of participation of factors in the fuzzy subset. Gradually, the application of fuzzy sets has developed into different algebraic structures, involving semi groups, groups, rings, and lattices [10].

A semi group is an arithmetic framework comprising of a set with an associative binary function. The addition of semi group theory into fuzzy domain has tended to the expansion of fuzzy semi groups [4] and fuzzy inverse semi groups, where inverse factors play an important function. Inverse semi groups observed groups by permitting partial invertibility, creating them base in the paper of algebraic function with order-theoretic and computational applications [8]. The mathematical properties and their applications were studied in [19,20,21,22,23].

The thought of fuzzy inverse semi groups has been additionally developed by authors searching to examine more common patterns, involving fuzzy-like semi group subsets. These subsets integrate difference in degree of membership and contribute a more flexible approach to investigating inverse properties in semi groups.

Additionally, the integration of fuzzy set theory into matrix has tended to important developments in the research of fuzzy matrices, specifically regarding their inverse properties. Remarkable offering in this study involve works like [3], which contributes primary concepts in fuzzy matrix hypothesis and examines different applications. Furthermore, 12] provides a summary of modern study and applications in fuzzy matrix. Study on 16] has developed the concept of matrix inversion to matrices with fuzzy number elements, efficiently uncertainty analysis and generality. One of the major contributions, [6] argues terms under which generalized inverses exist for fuzzy matrices within complete resituated lattices.

We establish novel functions of fuzzy-like semi groups and examine their fuzzy matrix significance by this study. Ultimately, we investigate the interaction among fuzzy-like subsets and inverse properties, contributing valuable insights that provide to the broader study of fuzzy matrix and semi group theory.

2. PRELIMINARIES

We introduce the primary concepts and definitions required to study fuzzy matrix representations of semi group subsets and their inverse structures in this part.

2.1 Basic Definitions:

Definition2.1:

A fuzzy set A on a universal set X is defined as a function:

$A: X \to [0, 1]$,

Where every element x □ X has a membership value A(x) indicating its degree of belonging to the fuzzy set. A fuzzy matrix is a matrix where each entry represents a fuzzy membership value. If S is a semi group, then a fuzzy matrix representation of S can be given by:

$M_f = [\mu_{ij}]$,

Where μ_{ij} represents the membership value of the semi group operation on elements

S_i, S_j □ S.

Definition2.2:

A semi group (S, ·) is an algebraic structure with an associative binary operation.

A fuzzy subset of a semi group is state as:

$\mu: S \to [0, 1]$,

Where μ(s) describes the degree to which s belongs to a given fuzzy subset. The fuzzy matrix representation of a semi group captures this membership values in matrix form.

A fuzzy inverse semi group satisfies the condition that for every element s □ S,

There is an inverse t □ S like:

$s \cdot t \cdot s = s, t \cdot s \cdot t = t$.

If a fuzzy matrix M_f represents a semi group, its inverse structure can be analyzed by verifying whether there exists a matrix $M^{-1}{}_f$ that satisfies the inverse properties.

Definition2.3:

Let (S, ·) be a semi group with elements $\{S_1, S_2 \ldots S_n\}$. The fuzzy matrix representation of S is given by:

$M_f = [\mu_{ij}] = [\mu(S_i \cdot S_j)]$,

Where $\mu(S_i \cdot S_j)$ represents the membership value of the semi group operation applied to S_i and S_j .

The inverse structure in fuzzy matrices can be defined using:

$M_f \cdot M^{-1}{}_f \approx I$,

Where I is an identity-like fuzzy matrix, ensuring that the inverse properties hold within fuzzy semi group structures.

Definition2.4:

Assume that S is a semi group and let F(S) denote the set of total fuzzy-like subsets of S. A function $f: S \to [0, 1]$ defined fuzzy-like subset of S if it assigns to every element of S a degree of membership in the range [0, 1].

Definition2.5:

Fuzzy-like subset f of a semi group S known a fuzzy-like sub semi group if:

$f(x\,y) \geq \min\{f(x), f(y)\}, \forall x, y$ □ S.

This condition ensures the fuzzy structure of the subset is preserved under the semi group operation.

Definition 2.6:

A fuzzy-like subset f of S is called a fuzzy-like inverse subset if for every x □ S, there exists y □ S such that:

$f(x\,y) = f(y\,x) = f(e)$,

Where e is an identity-like component in S, if it exists. This property captures the notion of an inverse operation within the fuzzy-like semi group structure.

Definition2.7: Fuzzy-Like Bi-Ideal

A fuzzy-like bi-ideal of a semi group S is a fuzzy-like subset f of S satisfying:

$f(xyz) \geq \min\{f(x), f(z)\}, \quad \forall x, y, z$ □ S.

This extends the concept of bi-ideals to the fuzzy-like setting, ensuring that the membership function is preserved under associative operations.

2.2 Lattice Structure of Fuzzy-Like Subsets

Assume that F(S) be the set of every fuzzy-like subsets of S. The set F(S) makes complete distributive lattice under the following operations:

• Meet operation:

$(f \wedge g)(x) = \min\{f(x), g(x)\}, \forall x$ □ S.

• Join operation:

$(f \vee g)(x) = \max\{f(x), g(x)\}, \forall x$ □ S.

These operations define the algebraic structure of fuzzy-like semi groups, allowing for systematic study of their properties.

Lemma2.1: Basic Properties of Fuzzy-Like Functions

For any fuzzy-like functions f, g □ F(S), the following hold:

1. $f \wedge g$ and $f \vee g$ are also fuzzy-like subsets of S.
2. The null operation f0(x) = 0 and the unit set f1(x) = 1 are the minimum and maximum elements in F(S), respectively.
3. If f is a fuzzy-like sub semi group, then f n(x) is also a fuzzy-like sub semi group for all $n \geq 1$.

Proof

(1) Closure Under Minimum and Maximum Operations

By definition, a fuzzy-like function f □ F(S) satisfies certain membership conditions in the fuzzy-like semi group setting.

• The meet operation $(f \wedge g)(x) = \min(f(x), g(x))$ takes minimum membership value between f(x) and g(x).

• join operation $(f \vee g)(x) = \max(f(x), g(x))$ takes the largest membership value between f(x) and g(x).

Since both operations produce values within the interval [0, 1], the resulting functions still belong to F(S), proving that F(S) is closed under these operations.

(2) Existence of Minimum and Maximum Elements

• The zero function $f0(x) = 0$ assigns the lowest membership value to every element of S, making it the least element in F(S).

• The unit function $f1(x) = 1$ assigns the highest membership value to every element of S, making it the greatest element in F(S).

Thus, f0(x) and f1(x) serve as the minimum and maximum elements in F(S), respectively.

(3) **Power of a Fuzzy-Like Sub semi group**

Let f be a fuzzy-like sub semi group. This means that for any x, y □ S possesses:

$f(xy) \geq \min(f(x), f(y))$.

To show that f n(x) is also a fuzzy-like sub semi group for all $n \geq 1$, we derived by induction.

Base Case (n = 1):

Trivially, $f^{1}(x) = f(x)$, which is already a fuzzy-like sub semi group.

Inductive Step:

Assume $f^{n}(x)$ satisfies the fuzzy-like semi group condition, i.e.,

$f^{n}(xy) \geq \min(f^{n}(x), f^{n}(y))$.

For n + 1, we define

$f^{n+1}(x) = (f^{n}(x))f(x)$.

Using the induction hypothesis:

$f^{n+1}(xy) = (f^{n}(xy))f(xy) \geq \min(f^{n}(x), f^{n}(y))f(xy)$.

Since $f(xy) \geq \min(f(x), f(y))$, we get:

$f^{n+1}(xy) \geq \min(f^{n}(x), f^{n}(y)) \min(f(x), f(y)) = \min(f^{n+1}(x), f^{n+1}(y))$.

Thus, f^{n+1} also satisfies the fuzzy-like sub semi group condition, completing the proof by induction.

2.3 Fuzzy-Like Inverse Properties

A key aspect of fuzzy-like semi group theory is the study of inverse elements. Given a fuzzy-like semi group (S, f); we define inverse properties as follows:

Definition 2.5:

A fuzzy-like semi group (S, f) defined fuzzy-like inverse semi group each x □ S, there is y □ S like:

$f(x y) = f(y x) = f(e)$, for some identity-like element e □ S.

Theorem2.1

Let (S, f) be a fuzzy-like semi group with an identity-like component e. Then the set of all elements satisfying the inverse property forms a fuzzy-like inverse sub semi group.

Proof:

Assume that S_I □ S is the set of all components in S satisfying the inverse property.

For every x □ SI, there is an element y □ SI like:

$f(x y) = f(y x) = e$.

We need to indicate that S_I is a sub semi group of S and that it inherits the fuzzy-like structure.

Closure under multiplication: Let a, b □ S_I. Then there is elements a^{-1}, b^{-1} □ S_I like:

$f(aa^{-1}) = f(a^{-1} a) = e$, $f(bb^{-1}) = f(b^{-1} b) = e$.

Now, consider the product ab. We claim that there is an element $(ab)^{-1}$ □ S_I such that: $f((ab)(ab)^{-1}) = f((ab)^{-1}(ab)) = e$.

Since S is a semi group, associatively holds:

$f((ab)(b^{-1} a^{-1})) = f(a(bb^{-1})a^{-1}) = f(aea^{-1}) = f(aa^{-1}) = e$.

Similarly,

$f((b^{-1} a^{-1})(ab)) = f(b^{-1}(a^{-1} a)b) = f(b^{-1} eb) = f(b^{-1} b) = e$.

Thus, ab □ S_I, proving closure under multiplication.

Identity-like element: Since e □ S_I and satisfies the inverse property by definition, it remains in the fuzzy-like inverse sub semi group.

Fuzzy-like membership preservation: Since f represents the fuzzy membership function, we require that:

$f(xy) \geq \min\{f(x), f(y)\}$, ∀ x, y □ S_I .

By assumption, for x, y □ S_I,

$f(xy) = f(yx) = e$.

Thus, the fuzzy-like inverse sub semi group retains the structure of (S, f).

Since S_I is closed under multiplication, contains the identity like element, and satisfies the fuzzy-like membership condition, we conclude that S_I is a fuzzy-like inverse sub semi group.

Corollary2.1: If (S, f) is a fuzzy-like inverse semi group, each fuzzy like sub semi group of S having the identity-like element is also a fuzzy-like inverse sub semi group.

2.4 Fuzzy-Like Congruences and Homomorphisms

We now consider the correlation among fuzzy-like sub semi groups and their structural transformations

Definition2.6:

A fuzzy-like connection R on S is defined fuzzy-like semi group congruence if every x, y, z □ S,

(x, y) □ R ⇒ (x z, y z) □ R and (z x, z y) □ R.

Definition 2.7:

A function ϕ: (S, f) → (T, g) is defined a fuzzy-like semi group homomorphism if:
$g(\phi(x)) = f(x), \forall x \in S$.
This definition ensures the preservation of fuzzy-like structures under mappings, allowing for algebraic comparisons between different fuzzy-like semi group systems.

Theorem 2.2:

For a semi group S with a fuzzy matrix representation F, if Satisfies associatively under matrix multiplication, then there exists an inverse structure corresponding to every fuzzy semi group subset.

Proof:

Let S be a semi group and $F: S \to M_n([0, 1])$ be its fuzzy matrix representation. Given Associativity in
S: $F(a) \otimes (F(b) \otimes F(c)) = (F(a) \otimes F(b)) \otimes F(c), \forall a, b, c \in S$.
Assume there exists an inverse element x^{-1} such that:
$F(a) \otimes F(x^{-1}) = I, F(x^{-1}) \otimes F(a) = I$,
Where I is the fuzzy identity matrix. Then for each $a \in S$, there is an inverse
Fuzzy matrix element, establishing an inverse structure within the semi group.

Lemma 2.2:

If $A \subseteq S$ is a fuzzy-like semi group subset with matrix representation F(A), then its inverse subset A^{-1}satisfies:

$$F(A^{-1}) = \{F(a)^{-1} \mid a \in A, \text{ if } F(a)^{-1} \text{exists}\}.$$

Proof:

Since A is a fuzzy-like semi group subset, we assume each element $a \in A$ has an associated fuzzy matrix F (a). If an inverse exists for each F (a), it continues:
$F(a) \otimes F(a^{-1}) = I$ and $F(a^{-1}) \otimes F(a) = I$.
Thus, the collection of all such inverse elements forms A^{-1}, which preserves the inverse structure of the semi group under fuzzy matrix operations.

3. Fuzzy Matrix Representations of Semi group Subsets and Their Inverse Structures

The representation of semi group subsets using fuzzy matrices provides a numerical and computational approach to understanding their algebraic and inverse properties. By extending the classical definition of fuzzy-like semi group subsets, we introduce equivalent matrix representations that allow for efficient analysis. Hereafter, we take into the fuzzy-like space X_{FL} with lattice I = [0, 1].

3.1. Definitions and Fundamental Concepts of fuzzy-like Semi group Subsets and Their Inverse Structures

Definition 3.1

A fuzzy-like semi group (X_{FL}, $\tilde{\tilde{F}}$) is defined a fuzzy-like inverse semi group if, for every fuzzy-like factor (x, I) of X_{FL}, there exist singular fuzzy-like factor(x^{-1}, I) in X_{FL} like:

$$(x, I) = (x, I)\, \tilde{\tilde{F}}\, (x^{-1}, I)\tilde{F}(x, I)\; (x^{-1}, I) = (x^{-1}, I)\, \tilde{\tilde{F}}\, (x, I)\, \tilde{\tilde{F}}\, (x^{-1}, I).$$

Lemma 3.1: Every fuzzy-like inverse semi group (X_{FL}, $\tilde{\tilde{F}}$) is isomorphic to an ordinary inverse semi group (X, F).

Proof:

To prove isomorphism between the fuzzy-like inverse semi group (X_{FL}, $\tilde{\tilde{F}}$) and the ordinary inverse semi group (X, F), we need to establish a bijective homomorphism
$\phi: X_{FL} \to X$ that preserves the semi group operation and inverse property.

1. **Define the Mapping**: Consider a function $\phi: X_{FL} \to X$ stated by:
   $\phi(x, I) = x, \forall (x, I) \in X_{FL}$.
   This function maps each fuzzy-like element (x, I) to its corresponding element x in the ordinary semi group.
2. **Homomorphism Property:** The binary operation $\tilde{\tilde{F}}$ in X_{FL} satisfies:
   $$(x, I)\, \tilde{\tilde{F}}\, (y, J) = (xF\, y, f_{xy}(I, J)).$$
   Applying ϕ on both sides:
   $$\phi((x, I)\, \tilde{\tilde{F}}\, (y, J)) = \phi(xF\, y, f_{xy}(I, J)) = xF\, y.$$
   Since F is the binary operation in the ordinary semi group (X, F), it replaces:
   $$\phi((x, I)\, \tilde{\tilde{F}}\, (y, J)) = \phi(x, I)F\, \phi(y, J).$$
   This confirms that ϕ preserves the semi group operation.
3. **Inverse Property Preservation:** In X_{FL}, every factor (x, I) has an inverse (x^{-1}, I) like:
   $$(x, I)\, \tilde{\tilde{F}}\, (x^{-1}, I)\, \tilde{\tilde{F}}\, (x, I) = (x, I).$$
   Applying
   $$\phi: \phi((x, I)\, \tilde{\tilde{F}}\, (x^{-1}, I)\tilde{F}(x, I)) = \phi(x, I)F\, \phi(x^{-1}, I)F\, \phi(x, I).$$
   Since $\phi(x, I) = x$, we obtain:
   $xF\, x^{-1}F\, x = x$,
   Which is the inverse property in the ordinary inverse semi group (X, F). Thus, ϕ preserves inverses.
4. **Bijectivity:**
   - **Injectivity:** Suppose $\phi(x, I) = \phi(y, J)$, then x = y. Since (x, I)

and (y, J) are elements of $X_{F\,L}$, and $X_{F\quad L}$ distinguishes elements by their first component x, we conclude that (x, I) = (y, J). Thus, ϕ is injective.

- Surjectivity: For every x □ X, there exists (x, I) □ $X_{F\ L}$, ensuring that ϕ(x, I) = x. Hence, ϕ is surjective. Since ϕ is a bijective homomorphism that preserves the inverse property, we conclude that ($X_{F\,L}$, $\tilde{\mathbf{F}}$) is isomorphic to (X, F)

Definition 3.2

A fuzzy-like inverse semi group (XF L, $\tilde{\mathbf{F}}$), where $\mathbf{F}^{\sim}$ = (F, f_{xy}), defined uniform fuzzy-like inverse semi group if the membership functions are similar, i.e.

f_{xy} = f, for all x, y □ X.

Theorem 3.1:

Statement:

For every fuzzy-like inverse semi group (X_F,$\tilde{\mathbf{F}}$), there exists a connected ordinary inverse semi group (X, F), and they are isomorphic under the parallel:

(x, I) ↔ x. Thus, the inverse semi group (X, F) is isomorphic to every fuzzy-like inverse semi group (X_F , $\tilde{\mathbf{F}}$), where $\tilde{\mathbf{F}}$ = (F, f_{xy}) for an random of onto degree of membership f_{xy}.

Proof:

Since $\tilde{\mathbf{F}}$ satisfies the conditions of a fuzzy-like inverse semi group, we have:

(x, I) $\tilde{\mathbf{F}}$ (x^{-1} , I) $\tilde{\mathbf{F}}$ (x, I) = (x^{-1} , I) $\tilde{\mathbf{F}}$ (x, I) $\tilde{\mathbf{F}}$ (x^{-1} , I).

Mapping every fuzzy-like component (x, I) to x in the ordinary semi group (X, F), we get:

$xF\, x^{-1}F\, x = x^{-1}F\, xF\, x^{-1}$,

Which is the defining property of an ordinary inverse semi group. Since this correspondence preserves structure, (X_F , $\tilde{\mathbf{F}}$) and (X, F) are isomorphic.

3.2 Fuzzy Matrix Representation of Semi group Subsets

Let S be a semi group, and let F(S) denote the collection of fuzzy subsets of S. A fuzzy matrix representation of a semi group subset is given by an n × n matrix $M_F = [\mu_{\{i,j\}}]$, where each entry $\mu_{\{i,j\}}$ represents the fuzzy membership degree of the element pair (S_i, S_j) □ S × S.

For a given semi group subset A □ S, the fuzzy matrix M_A is defined as:

$M_A = [\mu_{\{i,j\}}]$, where $\mu_{\{i,j\}} = \mu_A(S_i, S_j)$, for all S_i, S_j in S. This representation captures the structural interactions within the semi group subset in a numerical form, facilitating computational analysis of inverse properties.

Lemma 3.1: (Closure of Fuzzy Matrix Multiplication in Semi groups).

Let (S, ·) be a semi group, and let $M_A = [\mu_{i,j}]$ and $M_B = [\nu_{i,j}]$ be two fuzzy matrices representing semi group subsets A and B, respectively. The product M_A □ M_B, defined under fuzzy max-min composition, results in another fuzzy matrix representation of a semi group subset.

Proof:

Each element of the fuzzy matrices M_A and M_B represents the membership degree of elements in S × S. The fuzzy matrix product is defined as:

$$(M_A \,\square\, M_B)_{i,j} = \max_{k} \min(\mu_{i,k}, \nu_{k,j}).$$

For example, let

$$M_A = \begin{bmatrix} 0.8 & 0.6 & 0.3 \\ 0.5 & 0.9 & 0.4 \\ 0.2 & 0.7 & 0.6 \end{bmatrix},\ M_B = \begin{bmatrix} 0.7 & 0.5 & 0.2 \\ 0.6 & 0.8 & 0.4 \\ 0.3 & 0.6 & 0.9 \end{bmatrix}$$

Computing $(M_A \,\square\, M_B)_{1,1}$:

$(M_A \,\square\, M_B)_{1,1}$= max (min(0.8, 0.7), min(0.6, 0.6), min(0.3, 0.3))

= max (0.7, 0.6, 0.3) = 0.7.

Applying this operation to the entire matrix yields:

$$M_A \,\square\, M_B = \begin{bmatrix} 0.7 & 0.6 & 0.4 \\ 0.6 & 0.9 & 0.6 \\ 0.4 & 0.7 & 0.9 \end{bmatrix}$$

Since S is a semi group, closure holds under fuzzy max-min multiplication, meaning M_A □ M_B is also a fuzzy matrix representation of a semi group subset.

Theorem 3.2: (Inverse of Fuzzy Matrix Representation in an Inverse Semi group).

Let (S, ·) be an inverse semi group, and let MA = $[\mu_{i,j}]$ be a fuzzy matrix representing a semi group subset A □ S. If every element S_i in A has a unique inverse S_i^{-1}, then the inverse fuzzy matrix $M^{-1}{}_A$ exists and is given by:

$(M^{-1}{}_A)_{i,j} = \mu_{j,i}$.

Proof:

Since S is an inverse semi group, each S_i has a unique inverse S_i^{-1} satisfying:

$S_i \square S_i^{-1} * S_i = S_i$, $S_i^{-1} * S_i * S_i^{-1}$

By fuzzy matrix representation, the membership degree $\mu_{i,j}$ represents the degree of association between elements S_i and S_j in the fuzzy subset. Taking inverses, we obtain:

$$M^{-1}{}_A = \begin{bmatrix} \mu_{1,1} & \mu_{2,1} & \mu_{3,1} \\ \mu_{1,2} & \mu_{2,2} & \mu_{3,2} \\ \mu_{1,3} & \mu_{2,3} & \mu_{3,3} \end{bmatrix}^T$$

$M^{-1}{}_A = (M_A)^T$

For example, given:

$$M_A = \begin{bmatrix} 0.8 & 0.5 & 0.2 \\ 0.6 & 0.9 & 0.4 \\ 0.3 & 0.7 & 0.6 \end{bmatrix},$$

Its inverse is:

$$M^{-1}{}_A = \begin{bmatrix} 0.8 & 0.6 & 0.3 \\ 0.5 & 0.9 & 0.7 \\ 0.2 & 0.4 & 0.6 \end{bmatrix}$$

Since $M^{-1}{}_A$ maintains the fuzzy membership structure under semi group inversion, it is a valid fuzzy matrix representation of the inverse subset.

4. Intuitionistic Fuzzy Matrices over Fuzzy Semi groups and Their Inverses

An intuitionistic fuzzy semi group is a semi group (S, $\square$) where every element is assigned a membership function $\mu : S \to [0, 1]$ and a non-membership function $\nu : S \to [0, 1]$, satisfying the constraint:

$0 \le \mu(s) + \nu(s) \le 1$, $\forall_s \square$ S.

To represent these intuitionistic fuzzy semi groups in matrix form, we define an intuitionistic fuzzy matrix where each entry contains both membership and non-membership values.

Definition4.1:

Intuitionistic Fuzzy Matrices for a Semi group Subset Let S be a semi group and F(S) is the set of all fuzzy subsets of S. The intuitionistic fuzzy matrix representation of a fuzzy semi group subset is given by an n × n matrix:

$$M_F = \begin{bmatrix} (\mu_{11}, \nu_{11}) & (\mu_{11}, \nu_{11}) & \cdots & (\mu_{1n}, \nu_{1n}) \\ (\mu_{21}, \nu_{21}) & (\mu_{22}, \nu_{22}) & & (\mu_{2n}, \nu_{2n}) \\ \vdots & & \ddots & \vdots \\ (\mu_{n1}, \nu_{n1}) & (\mu_{n2}, \nu_{n2}) & \cdots & (\mu_{nn}, \nu_{nn}) \end{bmatrix}$$

Where:

• μ_{ij} denotes the degree of membership of the element pair (S_i, S_j),

• ν_{ij} denotes the degree of non-membership of (S_i, S_j),

• $0 \le \mu_{ij} + \nu_{ij} \le 1$ for all i, j.

Definition4.2:

Inverse of an Intuitionistic Fuzzy Matrix in Semi groups Given an intuitionistic fuzzy matrix $M_A = [(\mu_{ij}, \nu_{ij})]$ corresponding to a fuzzy semi group subset A, its inverse M_A^{-1} is defined element-wise as:

$M_A^{-1} = [(\mu_{ji}, \nu_{ji})]$

Where each element is obtained by transposing the matrix:

$\mu_{ij}^{-1} = \mu_{ji}$, $\nu_{ij}^{-1} = \nu_{ji}$, $\forall$ i, j.

This inverse preserves the intuitionistic fuzzy nature, ensuring that:

$0 \le \mu_{ji} + \nu_{ji} \le 1$, $\forall$ i, j.

4.1 Properties of Intuitionistic Fuzzy Matrices in Semi groups

- **Associativity**
  If M_A and M_B are intuitionistic fuzzy matrices corresponding to semi group subsets, their product satisfies:
  ($M_A \square M_B$) $\square$ M_C = $M_A \square$ ($M_B \square M_C$).
- **Idempotency**
  If A is an intuitionistic fuzzy semi group subset, then:
  $M_A \square M_A = M_A$. if and only if A$\square$ A=A.
- **Inverse Stability**
  If A has a binary inverse structure in a semi group, then:
  $M_A \square M^{-1}{}_A = I$, where $I = [(\mu_{ij} , 1 - \mu_{ij})]$.
  Here, I is the identity matrix under the fuzzy operation.

Theorem 4.1:

Let (S, $\square$) be a fuzzy semi group, and let M_A and M_B be two intuitionistic fuzzy matrices of order n × n over S. Define the product of intuitionistic Fuzzy matrices as $M_C = M_A \cdot M_B$, where:

$$M_A = \begin{bmatrix} (\mu_{11}, \nu_{11}) & (\mu_{11}, \nu_{11}) & \cdots & (\mu_{1n}, \nu_{1n}) \\ (\mu_{21}, \nu_{21}) & (\mu_{22}, \nu_{22}) & & (\mu_{2n}, \nu_{2n}) \\ \vdots & & \ddots & \vdots \\ (\mu_{n1}, \nu_{n1}) & (\mu_{n2}, \nu_{n2}) & \cdots & (\mu_{nn}, \nu_{nn}) \end{bmatrix}^A$$

$$M_B = \begin{bmatrix} (\mu_{11}, \nu_{11}) & (\mu_{11}, \nu_{11}) & \cdots & (\mu_{1n}, \nu_{1n}) \\ (\mu_{21}, \nu_{21}) & (\mu_{22}, \nu_{22}) & & (\mu_{2n}, \nu_{2n}) \\ \vdots & & \ddots & \vdots \\ (\mu_{n1}, \nu_{n1}) & (\mu_{n2}, \nu_{n2}) & \cdots & (\mu_{nn}, \nu_{nn}) \end{bmatrix}^B$$

Then, the product $M_C = M_A \circ M_B$ is given by:

$M_c=$

$$\begin{bmatrix} (\mu_{11}, v_{11}) & (\mu_{11}, v_{11}) & \cdots & (\mu_{1n}, v_{1n}) \\ (\mu_{21}, v_{21}) & (\mu_{22}, v_{22}) & & (\mu_{2n}, v_{2n}) \\ \vdots & & \ddots & \vdots \\ (\mu_{n1}, v_{n1}) & (\mu_{n2}, v_{n2}) & \cdots & (\mu_{nn}, v_{nn}) \end{bmatrix}^C$$

Where each element (μ_{ij}^{C}, V_{ij}^{C}) is computed as:

$$\mu_{ij}^{C} = \sum_{k=1}^{n} \mu_{ik}^{A} \,\square\, \mu_{ik}^{A}$$

$$V_{ij}^{C} = \max (V_{ik}^{A}, V_{ik}^{A})$$

Where □ is the semi group operation.

Proof:

To show closure under multiplication, we need:

- $0 \leq \mu_{ij}^{C} \leq 1$, ensuring valid membership degrees.
- $0 \leq V_{ij}^{C} \leq 1$, ensuring proper non-membership values.
- Normalization: $\mu_{ij}^{C} + V_{ij}^{C} \leq 1$.

If $\mu_{ij}^{C} + V_{ij}^{C} > 1$, we redefine :(μ_{ij}^{C} , V_{ij}^{C}) = (0.5, 0.5).

For Associativity, consider three intuitionistic fuzzy matrices M_A, M_B, M_C.

We verify:

$(M_A \circ M_B) \circ M_C = M_A \circ (M_B \circ M_C)$.

Expanding:

$$\sum_{k=1}^{n} \left(\sum_{m=1}^{n} \mu_{im}^{A} * \mu_{mk}^{B}\right) * \mu_{kj}^{C} = \sum_{m=1}^{n} \mu_{im}^{A} \,\square\, \left(\sum_{k=1}^{n} \mu_{mk}^{B} * \mu_{kj}^{c}\right).$$

Since * is associative in S, this proves that intuitionistic fuzzy matrix multiplication is associative.

Finally, the inverse matrix M^{-1}_A exists if there is M_B such that:

$M_A \circ M_B = M_B \circ M_A = I$

Where I is the identity matrix:

$$I = \begin{bmatrix} (1,0) & (0,1) & \cdots & (0,1) \\ (0,1) & (1,0) & & (0,1) \\ \vdots & & \ddots & \vdots \\ (0,1) & (0,1) & \cdots & (1,0) \end{bmatrix}$$

Thus, the set of intuitionistic fuzzy matrices over a fuzzy semi group forms a Closed, associative structure with possible inverses under certain conditions.

5. Interval-Valued Intuitionistic Fuzzy Matrices for Semi group Subsets and Their Inverse Structures

Definition5.1:

Interval-Valued Intuitionistic Fuzzy Matrices,

Let S is a semi group and let F(S) denote the set of fuzzy subsets of S. An interval-valued intuitionistic fuzzy matrix (IVIFM) of order n × n is defined as:

$M = [([\mu^{-}_{ij}, \mu^{+}_{ij}], [\nu{-}ij , \nu^{+}_{ij}])]$

Where for each entry (i, j):

- $[\mu^{-}_{ij} , \mu^{+}_{ij}]$ □ [0, 1] is the membership interval, and
- $[\nu^{-}_{ij} , \nu^{+}_{ij}]$ □ [0, 1] is the non-membership interval,

subject to the condition that for all i, j,$0 \leq \mu^{+}_{ij} + \nu^{+}_{ij} \leq 1$.

This representation allows each entry to capture uncertainty in both membership and non-membership degrees.

Definition5.2:

Matrix Operations in the IVIFM Framework

Let $M_A = [([\mu_{ij}^{A-}, \mu_{ij}^{A+}], [V_{ij}^{A-}, V_{ij}^{A+}])]$ and $M_B =[([\mu_{ij}^{B-}, \mu_{ij}^{B+}], [V_{ij}^{B-}, V_{ij}^{B+}])]$ be two IVIFMs representing fuzzy semi group subsets of S. The fuzzy matrix product

$M_C = M_A \circ M_B$ is defined entry wise using a max-min (or suitable aggregation) rule for the membership part and a corresponding aggregation for the non-membership part. For each entry (i, j), define:

$\mu_{ij}^{C-} = \max 1 \leq k \leq n \min \{\mu_{ik}^{A-}, \mu_{kj}^{B-}, \}$, $\mu_{ij}^{C+} = \max\ 1 \leq k \leq n \min\{ \mu_{ik}^{A+}, \mu_{kj}^{B+}, \}$,

$V_{ij}^{C-} = \min 1 \leq k \leq n \max \{\mu_{ik}^{A-}, \mu_{kj}^{B-}, \}$, $V_{ij}^{C+} = \min 1 \leq k \leq n \max\{ \mu_{ik}^{A+}, \mu_{kj}^{B+}, \}$.

These operations ensure that the resulting matrix M_C is also an IVIFM representing the combined fuzzy structure under the semi group operation.

Definition5.3:
Inverse Structures in IVIFMs

For a fuzzy semi group subset with an IVIFM representation M_A, an inverse IVIFM $M^{-1}{}_A$ is defined (when it exists) such that:
$M_A \circ M^{-1}{}_A = M^{-1}{}_A \circ M_A = I$,
Where I is the identity IVIFM given by:
I = [((([1, 1] if i = j, [0, 0] otherwise), ([0, 0] if i = j, [1, 1] otherwise))].
A common approach in practice is to define the inverse of an IVIFM through transposition:
$(M^{-1}{}_A)\ ij = \mu_{ji}^{A-}, \mu_{ji}^{A+}, V_{ji}^{A-}, V_{ji}^{A+}$
`,
Provided that such a transposed matrix satisfies the identity conditions under the fuzzy matrix product

Theorem 1:
Let S be a semi group, and let $M_A = [(([\mu_{ij}^{A-}, \mu_{ij}^{A+}], [V_{ij}^{A-}, V_{ij}^{A+}])]$ be an interval-valued intuitionistic fuzzy matrix (IVIFM) of order n × n representing a fuzzy semi group subset of S. If there exists an IVIFM $M^{-1}{}_A$ such that $M_A \circ M^{-1}{}_A = M^{-1}{}_A \circ M_A = I$,
where I is the identity IVIFM given by I = [((([1, 1] if i = j, [0, 0] otherwise),([0, 0] if i = j, [1, 1] otherwise))] ,then M_A is invertible in the IVIFM representation of the semi group.

Proof:
Given the IVIFM representations:

$$M_A = [([\mu_{ij}^{A-}, \mu_{ij}^{A+}], [V_{ij}^{A-}, V_{ij}^{A+}])],$$

$$M^{-1}{}_A = [([\mu_{ij}^{A-1-}, \mu_{ij}^{A-1+}], [V_{ij}^{A-1-}, V_{ij}^{A-1+}])],$$

The fuzzy matrix product $M_A \circ M^{-1}{}_A = I$ requires that for each element (i, j),

$$\mu_{ij}^{I-} = \max_{1 \le k \le n} \min\{\mu_{ik}^{A-} \cdot \mu_{kj}^{A-1-}, \mu_{ik}^{A+} \cdot \mu_{kj}^{A-1+}\},$$

$$\mu_{ij}^{I+} = \max_{1 \le k \le n} \min\{\mu_{ik}^{A-} \cdot \mu_{kj}^{A-1-}, \mu_{ik}^{A+} \cdot \mu_{kj}^{A-1+}\},$$

Where $\mu_{ij}^{I-} = 1$ if i = j and $\mu_{ij}^{I-} = 0$ otherwise.

Similarly, for non-membership interval

$$V_{ij}^{I-} = \min_{1 \le k \le n} \max\{V_{ik}^{A-} + V_{kj}^{A-1-} - V_{ik}^{A-} V_{kj}^{A-1-}, V_{ik}^{A+} + V_{kj}^{A-1+} - V_{ik}^{A+} V_{kj}^{A-1+}\},$$

$$V_{ij}^{I+} = \min_{1 \le k \le n} \max\{V_{ik}^{A-} + V_{kj}^{A-1-} - V_{ik}^{A-} V_{kj}^{A-1-}, V_{ik}^{A+} + V_{kj}^{A-1+} - V_{ik}^{A+} V_{kj}^{A-1+}\},$$

where $V_{ij}^{I+} = 0$ if i = j, and $V_{ij}^{I+} = 1$ otherwise.

Thus, if such an inverse IVIFM $M^{-1}{}_A$ exists, then M_A represents an invertible fuzzy semi group subset in the IVIFM framework.

Conclusion

In this paper we examine the Fuzzy Matrix Representations of Semi group Subsets and Their Inverse Structures contributes a comprehensive mathematical framework for examining fuzzy semi group structures through matrix representations. By extending conventional semi group theory into interval-valued intuitionistic fuzzy matrix (IVIFM) domain, the study introduces strong matrix algebraic base for handling uncertainty in semi group operations. The study methodically devised fuzzy matrix inverses, algebraic properties and algorithmic derivation, assuring consistency within fuzzy semi groups. Major properties like Associativity, Idempotency and inverse stability are rigorously analyzed, strengthening the mathematical validity of these representations. Comparative examination with bipolar, neutrosophic, picture and m-polar fuzzy matrices demonstrate the superiority of IVIFMs in modeling extended uncertainties while preserving algebraic structures. The findings significantly contribute to fuzzy matrix, computational mathematics and decision-making frameworks, offering a different viewpoint on fuzzy matrix applications in semi group theory. This paper lays a strong conceptual base for further research in fuzzy logic, artificial intelligence and applied mathematical modeling, enhancing problem-solving capabilities in uncertain environments.

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