New Perspectives on Fuzzy 2-Absorbing Ideals in Non-Commutative Rings

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.**Abstract**

In this work, we define and explore the concept of fuzzy 2-absorbing ideals within the framework of non-commutative rings. This notion serves as a generalization of both fuzzy ideals and the classical theory of 2-absorbing ideals to algebraic structures where the commutative property is not required. The results obtained extend several existing characterizations from commutative ring theory and enhance the study of fuzzy ideal structures in non-commutative algebraic systems.

Keywords: Fuzzy 2-Absorbing Ideal, Fuzzy Right Ideal, Fuzzy Left Ideal.
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1.Introduction

The concept of fuzzy sets was first introduced by Lotfi A. Zadeh in 1965 and has since become an important tool for dealing with uncertainty and gradual membership in mathematical structures. This theory has been widely incorporated into different areas of algebra, giving rise to the field known as fuzzy algebra. Among these developments, fuzzy ideals in ring theory have gained significant interest because they extend the traditional concept of ideals and allow elements to belong to an ideal with varying degrees of membership.

Badawi [1] established that a non-zero proper ideal I of R is a2-absorbing ideal. Gronewald [2] classified2 absorbing ideals into two classes namely 2 absorbing and strong 2absorbing. He established that the intersection of two prime ideals isa 2 absorbing ideals. He characterized that 2 absorbing ideals by the product of two left ideals. Serkan Onar, Deniz Sönmez, Bayram Ali Ersoy, Gürsel Yeşilot & Kostaq Hila [3] studied about Fuzzy 2-absorbing Primary ideals. Also, Serkan Onar [4] discussed about On Fuzzy 2-absorbing ideals in rings.

The concept of prime and semiprime ideals plays a central role in ring theory. Motivated by these classical notions, Badawi introduced the concept of 2-absorbing ideals as a natural generalization of prime ideals. This notion has since been extensively studied and extended to various algebraic settings.

In the context of non-commutative rings, additional difficulties arise because multiplication is not symmetric, requiring clear distinctions among left ideals, right ideals, and two-sided ideals. When fuzzy concepts are incorporated, these complexities become even more significant, since the sup–min composition used in defining the product of fuzzy subsets interacts in a complex way with the non-commutative nature of multiplication. Some applications were presented in [5-9].

2.Preliminaries:

Definition: 2.1

A fuzzy subset of R is a function $\mu:R \to [0,1]$ where $\mu(x)$ measures the degree of membership of x.

Definition: 2.2

Considering the ring is non-commutative, we distinguish left, right, and two-sided fuzzy ideals.

Fuzzy left ideal:

A fuzzy set $\tilde{\mu}$ on R is a fuzzy left ideal if for all $\tilde{x}, \tilde{y}, \tilde{r} \in R$ It should have additive closure i.e. $\tilde{\mu}(\tilde{x}-\tilde{y}) \geq \min\{\tilde{\mu}(\tilde{x}), \tilde{\mu}(\tilde{y})\}$ and the left absorption $\tilde{\mu}(\tilde{r}\tilde{x}) \geq \tilde{\mu}(\tilde{x})$.

Fuzzy right ideal:

A fuzzy set $\tilde{\mu}$ on $\dot{R}$ is a fuzzy right ideal if for all $\tilde{x}, \tilde{y}, \tilde{r} \in R$. It should have additive closure i.e. $\tilde{\mu}(\tilde{x}-\tilde{y}) \geq \min\{\tilde{\mu}(\tilde{x}), \tilde{\mu}(\tilde{y})\}$ and the right absorption $\tilde{\mu}(\tilde{x}\tilde{r}) \geq \tilde{\mu}(\tilde{x})$.

Definition:2.3

The fuzzy set $\tilde{\mu}$ is called a fuzzy 2-absorbing ideal of R if: $\tilde{\mu}$ is a fuzzy ideal of R, and for all a, b, c $\in$ R, $\tilde{\mu}(abc) \geq \min\{\tilde{\mu}(ab), \tilde{\mu}(ac), \tilde{\mu}(bc)\}$.

3.Some Theorems on Fuzzy 2-Absorbing Ideals in Non-Commutative rings

Theorem: 3.1

Let $\widetilde{A}$ be a fuzzy 2 absorbing ideal of R and let $\tilde{I}$ be a fuzzy left ideal of R. If $\tilde{I}RbRc \subseteq \widetilde{A}$ implies $bc \in \widetilde{A}$ or $\tilde{I}b \subseteq \widetilde{A}$ or $\tilde{I}c \subseteq \widetilde{A}$ for any left ideal $\tilde{I}$.

Proof:

Assume that $\tilde{I}RbRc \subseteq \widetilde{A}$,

suppose, to the contrary that

$$bc \notin \widetilde{A}, \tilde{I}b \not\subseteq \widetilde{A}, \tilde{I}c \not\subseteq \widetilde{A}.$$

Since $\tilde{I}b \not\subseteq \widetilde{A}$, there exists $x \in R$ such that

$\tilde{I}(x) > \widetilde{A}(xb)$ ---- (i)

Similarly, since $\tilde{I}c \not\subseteq \widetilde{A}$, there exists $y \in R$ such that

$\tilde{I}(y) > \tilde{A}(yc)$ ----- (ii)
Because $\tilde{I}RbRc \subseteq \tilde{A}$, we have
$\tilde{A}(xbc) \geq \tilde{I}(x)$, $\tilde{A}(ybc) \geq \tilde{I}(y)$----- (iii)
Now considering the element xbc. Since $\tilde{A}$ is a fuzzy 2 – absorbing ideal, we have
$\tilde{A}(xbc) \geq \min\{\tilde{A}(xb), \tilde{A}(bc), \tilde{A}(xc)\}$ --- (iv)
From our assumption $\tilde{A}(bc) < \tilde{I}(x)$, $\tilde{A}(xb) < \tilde{I}(x)$
Hence
$\min\{\tilde{A}(xb), \tilde{A}(bc), \tilde{A}(xc)\} < \tilde{I}(x)$ --- (v)
From (iv) and (v), it follows that
$\tilde{A}(xbc) < \tilde{I}(x)$
This contradicts (iii), which states that
$\tilde{A}(xbc) \geq \tilde{I}(x)$
Thus, our assumption is false. Therefore, at least one of the following holds
$bc \in \tilde{A}$ or $\tilde{I}b \subseteq \tilde{A}$ or $\tilde{I}c \subseteq \tilde{A}$
Hence the theorem proved.

Theorem: 3.2
Let $\tilde{A}$ be a fuzzy 2 absorbing ideal of R and let $\tilde{J}$ be a fuzzy left ideal of R.If a,c ∈ R implies $aR\tilde{J}Rc \subseteq \tilde{A}$ then $a\tilde{J} \subseteq \tilde{A}$ or $\tilde{J}c \subseteq \tilde{A}$ or $ac \in \tilde{A}$ for any left ideal $\tilde{J}$.

Proof:
Assume that $aR\tilde{J}Rc \subseteq \tilde{A}$.
Suppose, for contradiction, that
$a\tilde{J} \nsubseteq \tilde{A}, \tilde{J}c \nsubseteq \tilde{A}, ac \notin \tilde{A}$.
Then there exists $j \in \tilde{J}$ such that
$\tilde{A}(aj) < \tilde{J}(j)$ and $\tilde{A}(jc) < \tilde{J}(j)$. --- (i)
By hypothesis $a\dot{R}\tilde{J}\dot{R}c \subseteq \tilde{A}$, in particular
$\tilde{A}(ajc) \geq \tilde{J}(j)$ --- (ii)
Since $\tilde{A}$ is s fuzzy 2 – absorbing ideal, we have
$\tilde{A}(ajc) \geq \min\{\tilde{A}(aj), \tilde{A}(ac), \tilde{A}(jc)\}$ --- (iii)
By assumption (i), and the fact that $ac \notin \tilde{A}$, we have
$\tilde{A}(aj) < \tilde{J}(j), \tilde{A}(ac) < \tilde{J}(j), \tilde{A}(jc) < \tilde{J}(j)$,
So
$\min\{\tilde{A}(aj), \tilde{A}(ac), \tilde{A}(jc)\} < \tilde{J}(j)$ --- (iv)
From (iii) and (iv) it follows that
$\tilde{A}(ajc) < \tilde{J}(j)$
Which contradicts (ii), which say $\tilde{A}(ajc) \geq \tilde{J}(j)$
This Contradiction shows that our assumption is false.
Hence $a\tilde{J} \subseteq \tilde{A}$ or $\tilde{J}c \subseteq \tilde{A}$ or $ac \in \tilde{A}$.

Theorem: 3.3
Let $\tilde{A}$ be a fuzzy 2 absorbing ideal of R and let $\tilde{K}$ be a fuzzy left ideal of R. If $aRbR\tilde{K} \subseteq \tilde{A}$ implies $ab \in \tilde{A}$ or $b\tilde{K} \subseteq \tilde{A}$ or $a\tilde{K} \subseteq \tilde{A}$ for any left ideal $\tilde{K}$.

Proof:
Assume that $aRbR\tilde{K} \subseteq \tilde{A}$,
suppose, to the contrary that
$ab \notin \tilde{A}, b\tilde{K} \nsubseteq \tilde{A}, a\tilde{K} \nsubseteq \tilde{A}$.
Since $b\tilde{K} \nsubseteq \tilde{A}$, there exists $k \in R$ such that
$\tilde{K}(k) > \tilde{A}(bk)$ ---- (i)
Similarly, since $a\tilde{K} \nsubseteq \tilde{A}$, there exists $k_1 \in R$ such that
$\tilde{K}(k_1) \geq \tilde{A}(ak_1)$ ----- (ii)
Because $aRbR\tilde{K} \subseteq \tilde{A}$, we have
$\tilde{A}(abk) \geq \tilde{K}(k)$, $\tilde{A}(abk_1) \geq \tilde{K}(k_1)$----- (iii)
Now considering the element abk. Since $\tilde{A}$ is a fuzzy 2 – absorbing ideal, we have
$\tilde{A}(abk) \geq \min\{\tilde{A}(ab), \tilde{A}(bk), \tilde{A}(ak)\}$ --- (iv)
From our assumption $\tilde{A}(ab) < \tilde{K}(k)$, $\tilde{A}(bk) < \tilde{K}(k)$
Hence
$\min\{\tilde{A}(ab), \tilde{A}(bk), \tilde{A}(ak)\} < \tilde{K}(k)$ --- (v)
From (iv) and (v), it follows that
$\tilde{A}(abk) < \tilde{K}(k)$
This contradicts (iii), which states that
$\tilde{A}(abk) \geq \tilde{K}(k)$
Thus, our assumption is false. Therefore, at least one of the following holds
$ab \in \tilde{A}$ or $b\tilde{K} \subseteq \tilde{A}$ or $a\tilde{K} \subseteq \tilde{A}$
Hence the theorem proved.

Theorem:3.4
Let $\tilde{A}$ be a fuzzy 2 absorbing ideal of R. If $\tilde{I}_1\tilde{I}_2\tilde{I}_3 \subseteq \tilde{A}$ for any fuzzy ideals $\tilde{I}_1\tilde{I}_2\tilde{I}_3$ of R, then $\tilde{I}_1\tilde{I}_2 \subseteq \tilde{A}$ or $\tilde{I}_2\tilde{I}_3 \subseteq \tilde{A}$ or $\tilde{I}_1\tilde{I}_3 \subseteq \tilde{A}$.

Proof:
Assume that $\tilde{I}_1\tilde{I}_2\tilde{I}_3 \subseteq \tilde{A}$
Suppose, for contradiction, that
$\tilde{I}_1\tilde{I}_2 \nsubseteq \tilde{A}$ or $\tilde{I}_2\tilde{I}_3 \nsubseteq \tilde{A}$ or $\tilde{I}_1\tilde{I}_3 \nsubseteq \tilde{A}$.
Since $\tilde{I}_1\tilde{I}_2 \nsubseteq \tilde{A}$, there exists $a, b \in R$ such that
$\tilde{I}_1(a) > 0, \tilde{I}_2(b) > 0$
and
$\tilde{A}(ab) < \min\{\tilde{I}_1(a), \tilde{I}_2(b)\}$ --- (i)
Similarly, since $\tilde{I}_2\tilde{I}_3 \nsubseteq \tilde{A}$, there exists $b, c \in R$ such that
$\tilde{A}(bc) < \min\{\tilde{I}_2(b), \tilde{I}_3(c)\}$ --- (ii)
Also, since $\tilde{I}_1\tilde{I}_3 \nsubseteq \tilde{A}$, there exists $a, c \in R$ such that
$\tilde{A}(ac) < \min\{\tilde{I}_1(a), \tilde{I}_3(c)\}$ --- (iii)
Because
$\tilde{I}_1\tilde{I}_2\tilde{I}_3 \subseteq \tilde{A}$
We have $\tilde{A}(abc) \geq \min\{\tilde{I}_1(a), \tilde{I}_2(b), \tilde{I}_3(c)\}$ --- (iv)
Since $\tilde{A}$ is a fuzzy 2 – absorbing ideal, we have
$\tilde{A}(abc) \geq \min\{\tilde{A}(ab), \tilde{A}(bc), \tilde{A}(ac)\}$ --- (v)
From (i), (ii) and (iii) , we obtain
$\tilde{A}(ab) < m, \tilde{A}(bc) < m, \tilde{A}(ac) < m$
Where $m = \min\{\tilde{I}_1(a), \tilde{I}_2(b), \tilde{I}_3(c)\}$
Hence $\min\{\tilde{A}(ab), \tilde{A}(bc), \tilde{A}(ac)\} < m$ --- (vi)
From (v) and (iv), it follows that
$\tilde{A}(abc) \geq m$
Which contradicts (iv). Thus, our assumption is false.
Therefore at least one of the following holds
$\tilde{I}_1\tilde{I}_2 \subseteq \tilde{A}$ or $\tilde{I}_2\tilde{I}_3 \subseteq \tilde{A}$ or $\tilde{I}_1\tilde{I}_3 \subseteq \tilde{A}$.
Hence the theorem is proved.

Theorem: 3.5
Let $\tilde{A}$ be a fuzzy right ideal of R then the following statements are equivalent: Statement:1 For any fuzzy right ideals $\tilde{I}, \tilde{J}, \tilde{K}$ of R , $\tilde{I}\tilde{J}\tilde{K} \subseteq \tilde{A}$ implies $\tilde{I}\tilde{J} \subseteq \tilde{A}$ or $\tilde{J}\tilde{K} \subseteq \tilde{A}$ or $\tilde{I}\tilde{K} \subseteq \tilde{A}$

Statement:2 For any fuzzy ideals $\tilde{I}_1\tilde{I}_2\tilde{I}_3$ of R, $\tilde{I}_1\tilde{I}_2\tilde{I}_3 \subseteq \widetilde{A}$ implies $\tilde{I}_1\tilde{I}_2 \subseteq \widetilde{A}$ or $\tilde{I}_2\tilde{I}_3 \subseteq \widetilde{A}$ or $\tilde{I}_1\tilde{I}_3 \subseteq \widetilde{A}$.

Proof:

Assume that the statement (statement :1) holds.

Let $\tilde{I}_1\tilde{I}_2\tilde{I}_3$ be any fuzzy right ideals of R, such that $\tilde{I}_1\tilde{I}_2\tilde{I}_3 \subseteq \widetilde{A}$

Define $\tilde{I} = \tilde{I}_1, \tilde{J} = \tilde{I}_2, \tilde{K} = \tilde{I}_3$

Then

$\tilde{I}\tilde{J}\tilde{K} \subseteq \widetilde{A}$

By statement (1), $\tilde{I}\tilde{J} \subseteq \widetilde{A}$ or $\tilde{J}\tilde{K} \subseteq \widetilde{A}$ or $\tilde{I}\tilde{K} \subseteq \widetilde{A}$

Substituting back, $\tilde{I}_1\tilde{I}_2 \subseteq \widetilde{A}$ or $\tilde{I}_2\tilde{I}_3 \subseteq \widetilde{A}$ or $\tilde{I}_1\tilde{I}_3 \subseteq \widetilde{A}$.

Thus statement (2) Holds.

Let $\tilde{I}, \tilde{J}, \widetilde{K}$ be fuzzy ideals of R such that

$\tilde{I}\tilde{J}\tilde{K} \subseteq \widetilde{A}$

Set, $\tilde{I}_1 = \tilde{I}, \tilde{I}_2 = \tilde{J}, \tilde{I}_3 = \widetilde{K}$

Then $\tilde{I}_1\tilde{I}_2\tilde{I}_3 \subseteq \widetilde{A}$

By statement (2),

$\tilde{I}_1\tilde{I}_2 \subseteq \widetilde{A}$ or $\tilde{I}_2\tilde{I}_3 \subseteq \widetilde{A}$ or $\tilde{I}_1\tilde{I}_3 \subseteq \widetilde{A}$.

Substituting back

$\tilde{I}\tilde{J} \subseteq \widetilde{A}$ or $\tilde{J}\tilde{K} \subseteq \widetilde{A}$ or $\tilde{I}\tilde{K} \subseteq \widetilde{A}$

Thus statement (1) Holds

Hence the proof.

Theorem:3.6

Let $\widetilde{A}$ be a fuzzy ideal of R then the following statements are equivalent:

Statement:1 For any fuzzy left ideal $\tilde{I}$ of R. If $\tilde{I}RbRc \subseteq \widetilde{A}$ implies $bc \in \widetilde{A}$ or $\tilde{I}b \subseteq \widetilde{A}$ or $\tilde{I}c \subseteq \widetilde{A}$

Statement:2 For any fuzzy left ideals $\tilde{I}, \tilde{J}$ of R. If $\tilde{I}R\tilde{J}Rc \subseteq \widetilde{A}$ implies $\tilde{I}\tilde{J} \subseteq \widetilde{A}$ or $\tilde{J}c \subseteq \widetilde{A}$ or $\tilde{I}c \subseteq \widetilde{A}$

Proof:

Assume that S1 Holds.

Let $\tilde{I}, \tilde{J}$ be a fuzzy left ideal of R such that

$\tilde{I}R\tilde{J}Rc \subseteq \widetilde{A}$ --- (i)

We must prove that

$\tilde{I}\tilde{J} \subseteq \widetilde{A}$ or $\tilde{J}c \subseteq \widetilde{A}$ or $\tilde{I}c \subseteq \widetilde{A}$ – (ii)

If $\tilde{J} = 0$ then $\tilde{I}\tilde{J} = 0 \subseteq \widetilde{A}$ and the result holds.

Assume $\tilde{J} \neq 0$

Then there exists $b \in R$ such that

$\tilde{J}(b) > 0$

Since

$\tilde{I}R\tilde{J}Rc \subseteq \widetilde{A}$ for the element b we obtain

$\tilde{I}RbRc \subseteq \widetilde{A}$ – (iii)

By S1, we obtain

$bc \in \widetilde{A}$ or $\tilde{I}b \subseteq \widetilde{A}$ or $\tilde{I}c \subseteq \widetilde{A}$ --- (iv)

Now consider the possibilities

Case (i)

If $\tilde{I}c \subseteq \widetilde{A}$ then (ii) holds

Case (ii)

If $\tilde{I}b \subseteq \widetilde{A}$ for any $x \in R$

$(\tilde{I}\tilde{J})(x) = \operatorname{Sup}_{x=ij} \min\{\tilde{I}(i)\tilde{J}(j)\}$

Since $\tilde{J}(b) > 0$ and $\tilde{I}b \subseteq \widetilde{A}$, we obtain

$(\tilde{I}\tilde{J})(x) \leq \widetilde{A}(x)$ for all $x \in R$

Hence $\tilde{I}\tilde{J} \subseteq \widetilde{A}$

Case (iii)

If $bc \in \widetilde{A}$ then because $\tilde{J}$ is a fuzzy left ideal and $\widetilde{A}$ is a fuzzy ideal , it follows that

$\tilde{J}c \subseteq \widetilde{A}$

Thus S2 holds.

In all cases,

$\tilde{I}\tilde{J} \subseteq \widetilde{A}$ or $\tilde{J}c \subseteq \widetilde{A}$ or $\tilde{I}c \subseteq \widetilde{A}$

Hence S2 holds

Assume that S2 Holds.

Let $\tilde{I}$ be a fuzzy left ideal and suppose

$\tilde{I}RbRc \subseteq \widetilde{A}$ -- (v)

Define a fuzzy left ideal $\tilde{J}$ generated by b

Then $\tilde{J} = Rb$

Hence $\tilde{I}R\tilde{J}Rc = \tilde{I}R(Rb)Rc = \tilde{I}RbRc \subseteq \widetilde{A}$ -- (vi)

By S2

$\tilde{I}\tilde{J} \subseteq \widetilde{A}$ or $\tilde{J}c \subseteq \widetilde{A}$ or $\tilde{I}c \subseteq \widetilde{A}$

Now observe that

$\tilde{I}\tilde{J} = \tilde{I}(Rb) = \tilde{I}b$

Thus $\tilde{I}b \subseteq \widetilde{A}$

Also, $\tilde{J}c = (Rb)c = R(bc)$

Hence $R(bc) \subseteq \widetilde{A}$

Which implies $bc \in \widetilde{A}$

Otherwise, $\tilde{I}c \subseteq \widetilde{A}$

Therefore $bc \in \widetilde{A}$ or $\tilde{I}b \subseteq \widetilde{A}$ or $\tilde{I}c \subseteq \widetilde{A}$

Hence S1 Holds.

Theorem:3.7

Let $\widetilde{A}$ be a fuzzy ideal of R then the following statements are equivalent:

Statement:1 For any fuzzy left ideal $\tilde{I}$ of R. If $\tilde{I}RbRc \subseteq \widetilde{A}$ implies $bc \in \widetilde{A}$ or $\tilde{I}b \subseteq \widetilde{A}$ or $\tilde{I}c \subseteq \widetilde{A}$

Statement:2 For any fuzzy left ideals $\tilde{I}, \widetilde{K}$ of R. If $\tilde{I}RbR\widetilde{K} \subseteq \widetilde{A}$ implies $\tilde{I}b \subseteq \widetilde{A}$ or $b\widetilde{K} \subseteq \widetilde{A}$ or $\tilde{I}\widetilde{K} \subseteq \widetilde{A}$.

Proof:

Assume that S1 Holds.

Let $\tilde{I}, \widetilde{K}$ be a fuzzy left ideal of R such that

$\tilde{I}RbR\widetilde{K} \subseteq \widetilde{A}$ --- (i)

We must prove that

$\tilde{I}b \subseteq \widetilde{A}$ or $b\widetilde{K} \subseteq \widetilde{A}$ or $\tilde{I}\widetilde{K} \subseteq \widetilde{A}$ – (ii)

If $\widetilde{K} = 0$ then $\tilde{I}\widetilde{K} = 0 \subseteq \widetilde{A}$ and the result holds.

Assume $\widetilde{K} \neq 0$

Then there exists $c \in R$ such that

$\widetilde{K}(c) > 0$

Since

$\tilde{I}RbR\widetilde{K} \subseteq \widetilde{A}$ for the element c we obtain

$\tilde{I}RbRc \subseteq \widetilde{A}$ – (iii)

By S1, we obtain

$bc \in \widetilde{A}$ or $\tilde{I}b \subseteq \widetilde{A}$ or $\tilde{I}c \subseteq \widetilde{A}$ --- (iv)

Now consider the possibilities

Case (i)

If $\tilde{I}b \subseteq \widetilde{A}$ then (ii) holds

Case (ii)

If $bc \in \widetilde{A}$

Then since $\widetilde{K}$ is a fuzzy left ideal and $\widetilde{A}$ is a fuzzy ideal, it follows that

$b\widetilde{K} \subseteq \widetilde{A}$

Thus S2 holds

Case (iii)

If $\tilde{I}c \subseteq \widetilde{A}$ then for any $x \in R$

$\big(\tilde{\mathrm{I}}\tilde{\mathrm{K}}\big)(\mathrm{x}) = \operatorname{Sup}_{x=ik} \min\{\tilde{\mathrm{I}}(i)\tilde{\mathrm{K}}(\mathrm{k})\}$
Since $\tilde{\mathrm{K}}(c) > 0$ and $\tilde{\mathrm{I}}c \subseteq \tilde{\mathrm{A}}$, we obtain
$(\tilde{\mathrm{I}}\tilde{\mathrm{K}})(x) \leq \tilde{\mathrm{A}}(x)$ for all $x \in \mathrm{R}$
Hence $\tilde{\mathrm{I}}\tilde{\mathrm{K}} \subseteq \tilde{\mathrm{A}}$
Thus in all cases S2 holds.
Assume that S2 Holds.
Let $\tilde{\mathrm{I}}$ be a fuzzy left ideal and suppose
$\tilde{\mathrm{I}}\mathrm{R}b\mathrm{Rc} \subseteq \tilde{\mathrm{A}}$ -- (v)
Define a fuzzy left ideal $\tilde{\mathrm{K}}$ generated by c
Then $\tilde{\mathrm{K}} = Rc$
Hence $\tilde{\mathrm{I}}\mathrm{RbR}\tilde{\mathrm{K}} = \tilde{\mathrm{I}}\mathrm{RbR(Rc)} = \tilde{\mathrm{I}}\mathrm{R}b\mathrm{Rc} \subseteq \tilde{\mathrm{A}}$ -- (vi)
By S2
$\tilde{\mathrm{I}}\mathrm{b} \subseteq \tilde{\mathrm{A}}$ or $\mathrm{b}\tilde{\mathrm{K}} \subseteq \tilde{\mathrm{A}}$ or $\tilde{\mathrm{I}}\tilde{\mathrm{K}} \subseteq \tilde{\mathrm{A}}$.
Now simplify theses terms
If $\tilde{\mathrm{I}}b \subseteq \tilde{\mathrm{A}}$ then the result follows.
If $b\tilde{\mathrm{K}} \subseteq \tilde{\mathrm{A}}$ then $b(Rc) = R(bc) \subseteq \tilde{\mathrm{A}}$
Which implies $\mathrm{bc} \in \tilde{\mathrm{A}}$
If $\tilde{\mathrm{I}}\tilde{\mathrm{K}} \subseteq \tilde{\mathrm{A}}$ then $\tilde{\mathrm{I}}(Rc) = \tilde{\mathrm{I}}c \subseteq \tilde{\mathrm{A}}$
Therefore $bc \in \tilde{\mathrm{A}}$ or $\tilde{\mathrm{I}}b \subseteq \tilde{\mathrm{A}}$ or $\tilde{\mathrm{I}}\mathrm{c} \subseteq \tilde{\mathrm{A}}$
Hence S1 Holds.

Theorem: 3.8

Let $\tilde{\mathrm{A}}$ be a fuzzy ideal of R then the following statements are equivalent: Statement:1 For any fuzzy left ideal $\tilde{\mathrm{J}}$ of R. If $\mathrm{aR}\tilde{\mathrm{J}}\mathrm{Rc} \subseteq \tilde{\mathrm{A}}$ implies $\mathrm{a}\tilde{\mathrm{J}} \subseteq \tilde{\mathrm{A}}$ or $\tilde{\mathrm{J}}\mathrm{c} \subseteq \tilde{\mathrm{A}}$ or $\mathrm{ac} \in \tilde{\mathrm{A}}$

Statement:2 For any fuzzy left ideals $\tilde{\mathrm{I}}, \tilde{J}$ of R. If $\tilde{\mathrm{I}}\mathrm{R}\tilde{\mathrm{J}}\mathrm{Rc} \subseteq \tilde{\mathrm{A}}$ implies $\tilde{\mathrm{I}}\tilde{\mathrm{J}} \subseteq \tilde{\mathrm{A}}$ or $\tilde{\mathrm{J}}c \subseteq \tilde{\mathrm{A}}$ or $\tilde{\mathrm{I}}c \subseteq \tilde{\mathrm{A}}$

Proof:

Assume that S1 Holds.
Let $\tilde{\mathrm{I}}, \tilde{\mathrm{J}}$ be a fuzzy left ideal of R such that
$\tilde{\mathrm{I}}\mathrm{R}\tilde{\mathrm{J}}\mathrm{Rc} \subseteq \tilde{\mathrm{A}}$ --- (i)
We must prove that
$\tilde{\mathrm{I}}\tilde{\mathrm{J}} \subseteq \tilde{\mathrm{A}}$ or $\tilde{\mathrm{J}}\mathrm{c} \subseteq \tilde{\mathrm{A}}$ or $\tilde{\mathrm{I}}\mathrm{c} \subseteq \tilde{\mathrm{A}}$ – (ii)
Case:1 $\tilde{\mathrm{I}} = 0$ then $\tilde{\mathrm{I}}\tilde{\mathrm{J}} = 0 \subseteq \tilde{\mathrm{A}}$ and the result holds.
Case:2 $\tilde{\mathrm{I}} \neq 0$
Then there exists $a \in \mathrm{R}$ such that
$\tilde{\mathrm{I}}(a) > 0$
From (i), considering this element a, we get
$\mathrm{aR}\tilde{\mathrm{J}}Rc \subseteq \tilde{\mathrm{A}}$ --- (iii)
By S1, we obtain
$\mathrm{a}\tilde{\mathrm{J}} \subseteq \tilde{\mathrm{A}}$ or $\tilde{\mathrm{J}}\mathrm{c} \subseteq \tilde{\mathrm{A}}$ or $\mathrm{ac} \in \tilde{\mathrm{A}}$ --- (iv)
Subcase: (i) $\tilde{\mathrm{J}}\mathrm{c} \subseteq \tilde{\mathrm{A}}$ then (ii) holds
Subcase: (ii) $\mathrm{a}\tilde{\mathrm{J}} \subseteq \tilde{\mathrm{A}}$
We show $\tilde{\mathrm{I}}\tilde{\mathrm{J}} \subseteq \tilde{\mathrm{A}}$
For any $x \in \mathrm{R}$
$(\tilde{\mathrm{I}}\tilde{\mathrm{J}})(\mathrm{x}) = \operatorname{Sup}_{x=ij} \min\{\tilde{\mathrm{I}}(i)\tilde{\mathrm{J}}(\mathrm{j})\}$
Since $\tilde{\mathrm{I}}(a) > 0$, contributions involving a appear in the sup
Given $a\tilde{\mathrm{J}} \subseteq \tilde{\mathrm{A}}$, we have
$\tilde{\mathrm{A}}(aj) \geq \min(\tilde{\mathrm{I}}(a), \tilde{\mathrm{J}}(j))$
Thus, $(\tilde{\mathrm{I}}\tilde{\mathrm{J}})(\mathrm{x}) \leq \tilde{\mathrm{A}}(x)$
Hence, $\tilde{\mathrm{I}}\tilde{\mathrm{J}} \subseteq \tilde{\mathrm{A}}$
Subcase: (iii) $\mathrm{ac} \in \tilde{\mathrm{A}}$
Since $\tilde{\mathrm{A}}$ be a fuzzy left ideal
Thus for all $\mathrm{r} \in \mathrm{R}$, $\mathrm{arc} \in \tilde{\mathrm{A}}$
Using this and $\tilde{\mathrm{I}}(a) >$, we get $\tilde{\mathrm{I}}\mathrm{c} \subseteq \tilde{\mathrm{A}}$
Thus in all cases (ii) holds
Hence S2 holds.
Assume that S2 holds
Let $\tilde{\mathrm{J}}$ be a fuzzy left ideal such that
$\mathrm{aR}\tilde{\mathrm{J}}\mathrm{R}c \subseteq \tilde{\mathrm{A}}$ --- (v)
we must prove that, $\mathrm{a}\tilde{\mathrm{J}} \subseteq \tilde{\mathrm{A}}$ or $\tilde{\mathrm{J}}\mathrm{c} \subseteq \tilde{\mathrm{A}}$ or $\mathrm{ac} \in \tilde{\mathrm{A}}$
Let $\tilde{\mathrm{I}}$ be a fuzzy left ideal generated by a
Then $\tilde{\mathrm{I}} = Ra$
So, $\tilde{\mathrm{I}}R\tilde{\mathrm{J}}Rc = (Ra)R\tilde{\mathrm{J}}Rc = aR\tilde{\mathrm{J}}Rc \subseteq \tilde{\mathrm{A}}$ --- (vi)
Apply S2, From (vi), $\tilde{\mathrm{I}}\tilde{\mathrm{J}} \subseteq \tilde{\mathrm{A}}$ or $\tilde{\mathrm{J}}c \subseteq \tilde{\mathrm{A}}$ or $\tilde{\mathrm{I}}c \subseteq \tilde{\mathrm{A}}$
If $\tilde{\mathrm{J}}c \subseteq \tilde{\mathrm{A}}$, done.
If $\tilde{\mathrm{I}}\tilde{\mathrm{J}} \subseteq \tilde{\mathrm{A}}$ then $(\mathrm{Ra})\,\tilde{\mathrm{J}} = \mathrm{a}\tilde{\mathrm{J}} \subseteq \tilde{\mathrm{A}}$
If $\tilde{\mathrm{I}}c \subseteq \tilde{\mathrm{A}}$, then $(\mathrm{Ra})\, c = R(\mathrm{ac}) \subseteq \tilde{\mathrm{A}}$
Which implies $\mathrm{ac} \in \tilde{\mathrm{A}}$
Thus, $\mathrm{a}\tilde{\mathrm{J}} \subseteq \tilde{\mathrm{A}}$ or $\tilde{\mathrm{J}}\mathrm{c} \subseteq \tilde{\mathrm{A}}$ or $\mathrm{ac} \in \tilde{\mathrm{A}}$
Hence S1 holds.

Theorem: 3.9

Let $\tilde{\mathrm{A}}$ be a fuzzy ideal of R then the following statements are equivalent: Statement:1 For any fuzzy left ideal $\tilde{\mathrm{K}}$ of R. If $\mathrm{aRbR}\tilde{\mathrm{K}} \subseteq \tilde{\mathrm{A}}$ implies $\mathrm{ab} \in \tilde{\mathrm{A}}$ or $\mathrm{b}\tilde{\mathrm{K}} \subseteq \tilde{\mathrm{A}}$ or $a\tilde{\mathrm{K}} \subseteq \tilde{\mathrm{A}}$

Statement:2 For any fuzzy left ideals $\tilde{\mathrm{J}}, \tilde{K}$ of R. If $\mathrm{aR}\tilde{\mathrm{J}}\mathrm{R}\tilde{K} \subseteq \tilde{\mathrm{A}}$ implies $\mathrm{a}\tilde{\mathrm{J}} \subseteq \tilde{\mathrm{A}}$ or $\tilde{\mathrm{J}}\tilde{K} \subseteq \tilde{\mathrm{A}}$ or $a\tilde{\mathrm{K}} \subseteq \tilde{\mathrm{A}}$

Proof:

Assume that S1 Holds.
Let $\tilde{\mathrm{J}}, \tilde{\mathrm{K}}$ be a fuzzy left ideal of R such that
$\mathrm{aR}\tilde{\mathrm{J}}\mathrm{R}\tilde{\mathrm{K}} \subseteq \tilde{\mathrm{A}}$ --- (i)
We must prove that
$\mathrm{a}\tilde{\mathrm{J}} \subseteq \tilde{\mathrm{A}}$ or $\tilde{\mathrm{J}}\tilde{K} \subseteq \tilde{\mathrm{A}}$ or $a\tilde{\mathrm{K}} \subseteq \tilde{\mathrm{A}}$ – (ii)
Case:1 $\tilde{\mathrm{J}} = 0$ then $\tilde{\mathrm{J}}\tilde{\mathrm{K}} = 0 \subseteq \tilde{\mathrm{A}}$ so (ii) holds.
Case:2 $\tilde{\mathrm{J}} \neq 0$
Then there exists $b \in \mathrm{R}$ such that
$\tilde{\mathrm{J}}(b) > 0$
From (i), by definition of fuzzy product
$\mathrm{aRb}R\tilde{\mathrm{K}} \subseteq \tilde{\mathrm{A}}$ --- (iii)
By applying S1 to (iii), we obtain
$\mathrm{ab} \in \tilde{\mathrm{A}}$ or $\mathrm{b}\tilde{\mathrm{K}} \subseteq \tilde{\mathrm{A}}$ or $a\tilde{\mathrm{K}} \subseteq \tilde{\mathrm{A}}$ --- (iv)
Subcase: (i) $a\tilde{\mathrm{K}} \subseteq \tilde{\mathrm{A}}$ then (ii) holds
Subcase: (ii) $\mathrm{b}\tilde{\mathrm{K}} \subseteq \tilde{\mathrm{A}}$
We show $\tilde{\mathrm{J}}\tilde{\mathrm{K}} \subseteq \tilde{\mathrm{A}}$
For any $x \in \mathrm{R}$
$\big(\tilde{\mathrm{J}}\tilde{\mathrm{K}}\big)(\mathrm{x}) = \operatorname{Sup}_{x=yz} \min\{\tilde{\mathrm{J}}(y)\tilde{\mathrm{K}}(\mathrm{z})\}$
Since $\tilde{\mathrm{J}} > 0$, for any decomposition x = yz, the supremum includes terms involving elements close to b, In particular , for any z, $\tilde{\mathrm{A}}(\mathrm{bz}) \geq \tilde{\mathrm{K}}(z)$
Using the fact that $\tilde{\mathrm{A}}$ is a fuzzy ideal,
We get
$\tilde{\mathrm{A}}(yz) \geq \min(\tilde{\mathrm{J}}(y), \tilde{\mathrm{K}}(z))$
Thus, $\big(\tilde{\mathrm{J}}\tilde{\mathrm{K}}\big)(\mathrm{x}) \leq \tilde{\mathrm{A}}(x)$
Hence, $\tilde{\mathrm{J}}\tilde{\mathrm{K}} \subseteq \tilde{\mathrm{A}}$
Subcase: (iii) $\mathrm{ab} \in \tilde{\mathrm{A}}$
We prove $a\tilde{\mathrm{J}} \subseteq \tilde{\mathrm{A}}$
For any $r \in \mathrm{R}$
$(\mathrm{ar})\mathrm{b} = a(rb)$
Since $\tilde{\mathrm{J}}$ is a fuzzy ideal, $\tilde{\mathrm{J}}(rb) \geq \tilde{\mathrm{J}}(b) > 0$

Using $\widetilde{A}$ is a fuzzy ideal, we have
$\widetilde{A}(a(rb)) \geq \widetilde{A}(ab)$
Thus $\widetilde{A}(ar) \geq \tilde{J}(r)$
Which implies $a\tilde{J} \subseteq \widetilde{A}$
Thus in all cases (ii) holds
Hence S2 holds.
Assume that S2 holds
Let $\widetilde{K}$ be a fuzzy left ideal such that
$aRbR\widetilde{K} \subseteq \widetilde{A}$ --- (v)
we must prove that, $ab \in \widetilde{A}$ or $b\widetilde{K} \subseteq \widetilde{A}$ or $a\widetilde{K} \subseteq \widetilde{A}$
Define a fuzzy left ideal $\tilde{J} = Rb$
Then $\tilde{J}$ is a fuzzy left ideal and $aR\tilde{J}R\widetilde{K} = aRbR\widetilde{K} \subseteq \widetilde{A}$ --- (vi)
Apply S2, From (vi), $a\tilde{J} \subseteq \widetilde{A}$ or $\tilde{J}\widetilde{K} \subseteq \widetilde{A}$ or $a\widetilde{K} \subseteq \widetilde{A}$
Subcase(i) $a\widetilde{K} \subseteq \widetilde{A}$, done.
Subcase (ii) $\tilde{J}\widetilde{K} \subseteq \widetilde{A}$
Since $\tilde{J} = Rb$ we have $\tilde{J}\widetilde{K} = (Rb)\,\widetilde{K} = b\widetilde{K} \subseteq \widetilde{A}$
$a\tilde{J} \subseteq \widetilde{A}$, then $a(Rb) = R(ab) \subseteq \widetilde{A} \Rightarrow ab \in \widetilde{A}$
Thus, $ab \in \widetilde{A}$ or $b\widetilde{K} \subseteq \widetilde{A}$ or $a\widetilde{K} \subseteq \widetilde{A}$
Hence S1 holds.

Theorem: 3.10

Let $\widetilde{A}$ be a fuzzy ideal of R then the following statements are equivalent: Statement:1 For any fuzzy left ideals $\tilde{I}, \tilde{J}$ of R. If $\tilde{I}R\tilde{J}Rc \subseteq \widetilde{A}$ implies $\tilde{I}\tilde{J} \subseteq \widetilde{A}$ or $\tilde{J}c \subseteq \widetilde{A}$ or $\tilde{I}c \subseteq \widetilde{A}$

Statement:2 For any fuzzy left ideals $\tilde{I}, \tilde{J}, \widetilde{K}$ of R. If $\tilde{I}\tilde{J}\widetilde{K} \subseteq \widetilde{A}$ implies $\tilde{I}\tilde{J} \subseteq \widetilde{A}$ or $\tilde{J}\widetilde{K} \subseteq \widetilde{A}$ or $\tilde{I}\widetilde{K} \subseteq \widetilde{A}$

Proof:

Assume that S1 Holds.
Let $\tilde{I}, \tilde{J}, \widetilde{K}$ be a fuzzy left ideal of R such that
$\tilde{I}\tilde{J}\widetilde{K} \subseteq \widetilde{A}$ --- (i)
We must prove that
$\tilde{I}\tilde{J} \subseteq \widetilde{A}$ or $\tilde{J}\widetilde{K} \subseteq \widetilde{A}$ or $\tilde{I}\widetilde{K} \subseteq \widetilde{A}$– (ii)
Case:1 $\widetilde{K} = 0$ then $\tilde{J}\widetilde{K} = 0 \subseteq \widetilde{A}$ so (ii) holds.
Case:2 $\widetilde{K} \neq 0$
Then there exists $c \in R$ such that
$\widetilde{K}(c) > 0$
From (i), by definition of fuzzy product
$\tilde{I}R\tilde{J}Rc \subseteq \widetilde{A}$ --- (iii)
By applying S1 to (iii), we obtain
$\tilde{I}\tilde{J} \subseteq \widetilde{A}$ or $\tilde{J}c \subseteq \widetilde{A}$ or $\tilde{I}c \subseteq \widetilde{A}$ --- (iv)
Subcase: (i) $\tilde{I}\tilde{J} \subseteq \widetilde{A}$ then (ii) holds
Subcase: (ii) $\tilde{J}c \subseteq \widetilde{A}$
We show $\tilde{J}\widetilde{K} \subseteq \widetilde{A}$
For any $x \in R$
$(\tilde{J}\widetilde{K})(x) = \sup_{x=yz} \min\{\tilde{J}(y), \widetilde{K}(z)\}$
Since $\widetilde{K}(c) > 0$, the supremum includes terms involving elements c
From $\tilde{J}c \subseteq \widetilde{A}$. we have
$\widetilde{A}(yc) \geq \tilde{J}(y)$
Using the fact that $\widetilde{A}$ is a fuzzy ideal,
We get
$\widetilde{A}(yz) \geq \min(\tilde{J}(y), \widetilde{K}(z))$
Thus, $(\tilde{J}\widetilde{K})(x) \leq \widetilde{A}(x)$
Hence, $\tilde{J}\widetilde{K} \subseteq \widetilde{A}$
Subcase: (iii) $\tilde{I}c \subseteq \widetilde{A}$
We prove $\tilde{I}\widetilde{K} \subseteq \widetilde{A}$
For any $x \in R$
$(\tilde{I}\widetilde{K})(x) = \sup_{x=yz} \min\{\tilde{I}(y), \widetilde{K}(z)\}$
Using $\tilde{I}c \subseteq \widetilde{A}$ and $\widetilde{K}c > 0$ we have
$(\tilde{I}\widetilde{K})(x) \leq \widetilde{A}(x)$
Thus $\tilde{I}\widetilde{K} \subseteq \widetilde{A}$
Thus in all cases (ii) holds
Hence S2 holds.
Assume that S2 holds
Let $\tilde{I}, \tilde{J}$ be a fuzzy left ideal such that
$\tilde{I}R\tilde{J}Rc \subseteq \widetilde{A}$ --- (v)
we must prove that, $\tilde{I}\tilde{J} \subseteq \widetilde{A}$ or $\tilde{J}c \subseteq \widetilde{A}$ or $\tilde{I}c \subseteq \widetilde{A}$
Define a fuzzy left ideal $\widetilde{K} = Rc$
Then $\widetilde{K}$ is a fuzzy left ideal also $\tilde{I}\tilde{J}\widetilde{K} = \tilde{I}\tilde{J}(Rc) = \tilde{I}R\tilde{J}Rc \subseteq \widetilde{A}$ --- (vi)
Apply S2, From (vi), $\tilde{I}\tilde{J} \subseteq \widetilde{A}$ or $\tilde{J}\widetilde{K} \subseteq \widetilde{A}$ or $\tilde{I}\widetilde{K} \subseteq \widetilde{A}$
Subcase(i) $\tilde{I}\tilde{J} \subseteq \widetilde{A}$, done.
Subcase (ii) $\tilde{J}\widetilde{K} \subseteq \widetilde{A}$
Since $\widetilde{K} = Rc$, $\tilde{J}\widetilde{K} = \tilde{J}\,(Rc) = \tilde{J}c \subseteq \widetilde{A}$
Subcase (iii) $\tilde{I}\widetilde{K} \subseteq \widetilde{A}$
Similarly $\tilde{I}\widetilde{K} = \tilde{I}c$
So $\tilde{I}c \subseteq \widetilde{A}$
Thus, $\tilde{I}\tilde{J} \subseteq \widetilde{A}$ or $\tilde{J}c \subseteq \widetilde{A}$ or $\tilde{I}c \subseteq \widetilde{A}$
Hence S1 holds.

Theorem: 3.11

Let $\widetilde{A}$ be a fuzzy ideal of R then the following statements are equivalent: Statement:1 For any fuzzy left ideals $\tilde{J}, \widetilde{K}$ of R. If $aR\tilde{J}R\widetilde{K} \subseteq \widetilde{A}$ implies $a\tilde{J} \subseteq \widetilde{A}$ or $\tilde{J}\widetilde{K} \subseteq \widetilde{A}$ or $a\widetilde{K} \subseteq \widetilde{A}$

Statement:2 For any fuzzy left ideals $\tilde{I}, \tilde{J}, \widetilde{K}$ of R. If $\tilde{I}\tilde{J}\widetilde{K} \subseteq \widetilde{A}$ implies $\tilde{I}\tilde{J} \subseteq \widetilde{A}$ or $\tilde{J}\widetilde{K} \subseteq \widetilde{A}$ or $\tilde{I}\widetilde{K} \subseteq \widetilde{A}$

Proof:

Assume that S1 Holds.
Let $\tilde{I}, \tilde{J}, \widetilde{K}$ be a fuzzy left ideal of R such that
$\tilde{I}\tilde{J}\widetilde{K} \subseteq \widetilde{A}$ --- (i)
We must prove that
$\tilde{I}\tilde{J} \subseteq \widetilde{A}$ or $\tilde{J}\widetilde{K} \subseteq \widetilde{A}$ or $\tilde{I}\widetilde{K} \subseteq \widetilde{A}$– (ii)
Case:1 $\widetilde{K} = 0$ then $\tilde{J}\widetilde{K} = 0 \subseteq \widetilde{A}$ so (ii) holds.
Case:2 $\widetilde{K} \neq 0$
Then there exists $c \in R$ such that
$\widetilde{K}(c) > 0$
From (i), by definition of fuzzy product
$(\tilde{I}\tilde{J}\widetilde{K})(x) = \sup_{x=uvw} \min\{\tilde{I}(u), \tilde{J}(v)\widetilde{K}(w)\} \leq \widetilde{A}$
Fix $a \in R$. Since $\tilde{I}$ is a fuzzy left ideal,
$a\tilde{I} \subseteq \tilde{I}$
Thus, $aR\tilde{J}R\widetilde{K} \subseteq \tilde{I}\tilde{J}\widetilde{K} \subseteq \widetilde{A}$
By applying S1 to (iii), we obtain
$a\tilde{J} \subseteq \widetilde{A}$ or $\tilde{J}\widetilde{K} \subseteq \widetilde{A}$ or $a\widetilde{K} \subseteq \widetilde{A}$ --- (iii)
Subcase: (i) $\tilde{J}\widetilde{K} \subseteq \widetilde{A}$ then (ii) holds
Subcase: (ii) $a\tilde{J} \subseteq \widetilde{A}$ for all $a \in R$, then

$\tilde{I}\tilde{J} \subseteq \widetilde{A}$
If $a\widetilde{K} \subseteq \widetilde{A}$ for all $a \in R$, then
$\tilde{I}\widetilde{K} \subseteq \widetilde{A}$
Hence (ii) holds.
Assume that S2 holds

Let $\tilde{J}$, $\tilde{K}$ be a fuzzy left ideal such that
$aR\tilde{J}R\tilde{K} \subseteq \tilde{A}$ --- (iv)
we must prove that, $a\tilde{J} \subseteq \tilde{A}$ or $\tilde{J}\tilde{K} \subseteq \tilde{A}$ or $a\tilde{K} \subseteq \tilde{A}$
Define a fuzzy left ideal $\tilde{I} = Ra$
Then $\tilde{I}\tilde{J}\tilde{K} = (Ra)\tilde{J}\tilde{K} \subseteq aR\tilde{J}R\tilde{K} \subseteq \tilde{A}$ --- (v)
Apply S2, From (v), $\tilde{I}\tilde{J} \subseteq \tilde{A}$ or $\tilde{J}\tilde{K} \subseteq \tilde{A}$ or $\tilde{I}\tilde{K} \subseteq \tilde{A}$
Subcase(i) $Ra\tilde{J} \subseteq \tilde{A} \Rightarrow a\tilde{J} \subseteq \tilde{A}$ or
Subcase (ii) $\tilde{J}\tilde{K} \subseteq \tilde{A}$ or
Subcase (iii) $Ra\tilde{K} \subseteq \tilde{A} \Rightarrow a\tilde{K} \subseteq \tilde{A}$
Thus, $a\tilde{J} \subseteq \tilde{A}$ or $\tilde{J}\tilde{K} \subseteq \tilde{A}$ or $a\tilde{K} \subseteq \tilde{A}$
Hence S1 holds.

Theorem:3.12

Let $\tilde{A}$ be a fuzzy ideal of R then the following statements are equivalent:
Statement:1 For any fuzzy left ideals $\tilde{I}, \tilde{K}$ of R. If $\tilde{I}RbR\tilde{K} \subseteq \tilde{A}$ implies $\tilde{I}b \subseteq \tilde{A}$ or $b\tilde{K} \subseteq \tilde{A}$ or $\tilde{I}\tilde{K} \subseteq \tilde{A}$.
Statement:2 For any fuzzy left ideals $\tilde{I}, \tilde{J}, \tilde{K}$ of R. If $\tilde{I}\tilde{J}\tilde{K} \subseteq \tilde{A}$ implies $\tilde{I}\tilde{J} \subseteq \tilde{A}$ or $\tilde{J}\tilde{K} \subseteq \tilde{A}$ or $\tilde{I}\tilde{K} \subseteq \tilde{A}$

Proof:

Assume that S1 Holds.
Let $\tilde{I}, \tilde{J}, \tilde{K}$ be a fuzzy left ideal of R such that
$\tilde{I}\tilde{J}\tilde{K} \subseteq \tilde{A}$ --- (i)
We must prove that
$\tilde{I}\tilde{J} \subseteq \tilde{A}$ or $\tilde{J}\tilde{K} \subseteq \tilde{A}$ or $\tilde{I}\tilde{K} \subseteq \tilde{A}$ --- (ii)
Case:1 $\tilde{J} = 0$ then $\tilde{I}\tilde{J} = 0 \subseteq \tilde{A}$ so (ii) holds.
Case:2 $\tilde{J} \neq 0$
Then there exists $b \in R$ such that
Since $\tilde{I}$ is a fuzzy left ideal,
$\tilde{I}R \subseteq \tilde{I}$
Hence, $\tilde{I}RbR\tilde{K} \subseteq \tilde{I}\tilde{J}\tilde{K} \subseteq \tilde{A}$ --- (iii)
By applying S1 to (iii), we obtain
$\tilde{I}b \subseteq \tilde{A}$ or $b\tilde{K} \subseteq \tilde{A}$ or $\tilde{I}\tilde{K} \subseteq \tilde{A}$ --- (iv)
Subcase: (i) $\tilde{I}\tilde{K} \subseteq \tilde{A}$ then (ii) holds
Subcase: (ii) $\tilde{I}b \subseteq \tilde{A}$ for all $b \in \tilde{J}$, then
$\tilde{I}\tilde{J} \subseteq \tilde{A}$
Subcase: (iii) $b\tilde{K} \subseteq \tilde{A}$ for all $b \in \tilde{J}$, then
$\tilde{J}\tilde{K} \subseteq \tilde{A}$
Hence (ii) holds.
Assume that S2 holds
Let $\tilde{I}$, $\tilde{K}$ be a fuzzy left ideal such that
$\tilde{I}RbR\tilde{K} \subseteq \tilde{A}$ --- (v)
we must prove that, $\tilde{I}b \subseteq \tilde{A}$ or $b\tilde{K} \subseteq \tilde{A}$ or $\tilde{I}\tilde{K} \subseteq \tilde{A}$
Define a fuzzy left ideal $\tilde{J} = RbR$
Then $\tilde{I}\tilde{J}\tilde{K} = \tilde{I}(RbR)\tilde{K} \subseteq \tilde{I}RbR\tilde{K} \subseteq \tilde{A}$ --- (vi)
Apply S2, From (vi), $\tilde{I}\tilde{J} \subseteq \tilde{A}$ or $\tilde{J}\tilde{K} \subseteq \tilde{A}$ or $\tilde{I}\tilde{K} \subseteq \tilde{A}$
Subcase(i) $\tilde{I}\tilde{J} = \tilde{I}(RbR) \subseteq \tilde{A} \Rightarrow \tilde{I}b \subseteq \tilde{A}$ or
Subcase (ii) $\tilde{J}\tilde{K} = (RbR)\tilde{K} \subseteq \tilde{A} \Rightarrow b\tilde{K} \subseteq \tilde{A}$
Similarly, $\tilde{I}\tilde{K} \subseteq \tilde{A}$
Thus, $\tilde{I}b \subseteq \tilde{A}$ or $b\tilde{K} \subseteq \tilde{A}$ or $\tilde{I}\tilde{K} \subseteq \tilde{A}$
Hence S1 holds.

Theorem:3.13

Let $\tilde{A}$ be a fuzzy ideal of R then the following statements are equivalent:
Statement:1 For any fuzzy left ideals $\tilde{I}, \tilde{J}, \tilde{K}$ of R. If $\tilde{I}\tilde{J}\tilde{K} \subseteq \tilde{A}$ implies $\tilde{I}\tilde{J} \subseteq \tilde{A}$ or $\tilde{J}\tilde{K} \subseteq \tilde{A}$ or $\tilde{I}\tilde{K} \subseteq \tilde{A}$.
Statement:2 For any fuzzy ideals $\tilde{I}_1\tilde{I}_2\tilde{I}_3$ of R. If If $\tilde{I}_1\tilde{I}_2\tilde{I}_3 \subseteq \tilde{A}$ implies $\tilde{I}_1\tilde{I}_2 \subseteq \tilde{A}$ or $\tilde{I}_2\tilde{I}_3 \subseteq \tilde{A}$ or $\tilde{I}_1\tilde{I}_3 \subseteq \tilde{A}$.

Proof

Assume that S1 Holds.
Let $\tilde{I}_1\tilde{I}_2\tilde{I}_3$ be a fuzzy left ideal of R such that
$\tilde{I}_1\tilde{I}_2\tilde{I}_3 \subseteq \tilde{A}$ --- (i)
We must prove that
$\tilde{I}_1\tilde{I}_2 \subseteq \tilde{A}$ or $\tilde{I}_2\tilde{I}_3 \subseteq \tilde{A}$ or $\tilde{I}_1\tilde{I}_3 \subseteq \tilde{A}$. --- (ii)
Since every fuzzy ideal is in particular, a fuzzy left ideal, $\tilde{I}_1\tilde{I}_2\tilde{I}_3$ are fuzzy left ideals
Thus from (i) $\tilde{I}_1\tilde{I}_2\tilde{I}_3 \subseteq \tilde{A}$
Applying S1, we get
$\tilde{I}_1\tilde{I}_2 \subseteq \tilde{A}$ or $\tilde{I}_2\tilde{I}_3 \subseteq \tilde{A}$ or $\tilde{I}_1\tilde{I}_3 \subseteq \tilde{A}$
Thus (ii) holds
Hence S2 proved.
Assume that S2 holds
Let $\tilde{I}, \tilde{J}, \tilde{K}$ be a fuzzy left ideal of R such that
$\tilde{I}\tilde{J}\tilde{K} \subseteq \tilde{A}$ --- (i)
We must prove that
$\tilde{I}\tilde{J} \subseteq \tilde{A}$ or $\tilde{J}\tilde{K} \subseteq \tilde{A}$ or $\tilde{I}\tilde{K} \subseteq \tilde{A}$ --- (ii)
Since $\tilde{A}$ is a fuzzy ideal, we can consider the fuzzy ideals generated by $\tilde{I}, \tilde{J}, \tilde{K}$
$\tilde{I}_1 = \tilde{I}$, $\tilde{I}_2 = \tilde{J}$, $\tilde{I}_3 = \tilde{K}$
Then $\tilde{I}_1\tilde{I}_2\tilde{I}_3 \subseteq \tilde{A}$
Applying S2, we get $\tilde{I}\tilde{J} \subseteq \tilde{A}$ or $\tilde{J}\tilde{K} \subseteq \tilde{A}$ or $\tilde{I}\tilde{K} \subseteq \tilde{A}$
Thus S1 holds.

Theorem:14

Let $\tilde{A}$ be a fuzzy 2 absorbing ideal of R and let $\tilde{I}_1\tilde{I}_2\tilde{I}_3$ be a fuzzy left ideal of R. If $\tilde{I}_1\tilde{I}_2\tilde{I}_3 \subseteq \tilde{A}$ implies $\tilde{I}_1\tilde{I}_2 \subseteq \tilde{A}$ or $\tilde{I}_2\tilde{I}_3 \subseteq \tilde{A}$ or $\tilde{I}_1\tilde{I}_3 \subseteq \tilde{A}$.

Proof:

Assume $\tilde{I}_1\tilde{I}_2\tilde{I}_3 \subseteq \tilde{A}$
This means that for all
$(\tilde{I}_1\tilde{I}_2\tilde{I}_3)(x) \leq \tilde{A}(x)$
Take arbitrary elements
$a \in \tilde{I}_1, b \in \tilde{I}_2, c \in \tilde{I}_3$
Then $abc \in \tilde{I}_1\tilde{I}_2\tilde{I}_3 \subseteq \tilde{A}$
So $abc \in \tilde{A}$
Since $\tilde{A}$ is fuzzy 2 – absorbing
$abc \in \tilde{A} \Rightarrow ab \in \tilde{A}$ or $bc \in \tilde{A}$ or $ac \in \tilde{A}$
Suppose the conclusion is false that is
$\tilde{I}_1\tilde{I}_2 \nsubseteq \tilde{A}$ then there exists $a_1 \in \tilde{I}_1$, $b_1 \in \tilde{I}_2$ such that $a_1\, b_1 \notin \tilde{A}$
$\tilde{I}_2\tilde{I}_3 \nsubseteq \tilde{A}$ then there exists $b_2 \in \tilde{I}_2$, $c_1 \in \tilde{I}_3$ such that $b_2\, c_1 \notin \tilde{A}$
$\tilde{I}_1\tilde{I}_3 \nsubseteq \tilde{A}$ then there exists $a_2 \in \tilde{I}_3$, $c_1 \in \tilde{I}_3$ such that $a_2\, c_2 \notin \tilde{A}$
Now consider
$a = a_2, b = b_2, c = c_2$
Then $a \in \tilde{I}_1, b \in \tilde{I}_2, c \in \tilde{I}_3$
So $abc \in \tilde{A}$
But $ab \notin \tilde{A}$, $bc \notin \tilde{A}$, $ac \notin \tilde{A}$
This contradicts the fuzzy 2- absorbing property
Therefore our assumption is false
Hence $\tilde{I}_1\tilde{I}_2 \subseteq \tilde{A}$ or $\tilde{I}_2\tilde{I}_3 \subseteq \tilde{A}$ or $\tilde{I}_1\tilde{I}_3 \subseteq \tilde{A}$.

Conclusion

The aim of this paper is to define and analyze fuzzy 2-absorbing ideals within the context of non-commutative rings. We explore their basic properties, provide characterizations using products of fuzzy ideals, and study their connections with fuzzy prime and fuzzy semiprime ideals. Various relations are included and proved to highlight the unique features of fuzzy 2-absorbing ideals in non-commutative environments.

This work extends the theory of fuzzy 2-absorbing ideals to a more general algebraic setting and enhances the understanding of fuzzy ideal structures in non-commutative rings, thereby creating opportunities for further developments in fuzzy algebra and its practical applications.

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