Introduction to Operations on q- fraction Hesitancy Fuzzy Graphs

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Abstract

Conventional fuzzy graph models, like intuitionistic fuzzy graphs (IFGs), Pythagorean fuzzy graphs (PyFGs), and q-rung orthopair fuzzy graphs (q-ROFGs), have become effective tools for simulating uncertainty in intricate networked structures in recent years. These models exhibit significant limitations when both membership and non-membership degrees go close to extreme values, despite their advancements. These limitations make it impossible for these frameworks to accurately simulate a wide range of real-world scenarios, where they are regularly broken. Furthermore, the underlying dependence between the membership, non-membership, and reluctance grades is frequently difficult for current models to remove.

I provide a novel structure, the q-Fractional hesitation Fuzzy Graph (q-FHFG), which expands fuzzy graph theory and permits more flexibility and independence between membership, non-membership, and hesitant grades in order to get around these restrictions. In contrast to earlier models, q-fractional hesitation fuzzy graphs allow both membership, non-membership, and hesitancy values to approach 1 independently without breaking consistency, allowing them to handle extremely high levels of uncertainty. The structure of q-fractional hesitancy fuzzy graphs and various operations on them are formally defined. The relationship is also discussed.

***Keywords**:* q-fractional Hesitancy fuzzy graphs, P-union and R-union.

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Introduction

A fascinating branch of mathematics, graph theory explores the intricate web of connections that form the basic structure of our universe. Graph theory is a fundamental tool that connects theoretical mathematics with practical applications, from social systems to communication networks, making it a crucial part of contemporary scientific and technological developments. A crucial change from the strict binary logic of classical set theory to a more adaptable framework that permits partial membership was made in 1965 when Zadeh [8] established the idea of a fuzzy set. Rosenfeld formalized fuzzy relations and fuzzy graphs after Kauffman first presented the idea of fuzzy graphs.

Expanding upon the notion of fuzzy set (FS), Atanassov [2] created an intuitionistic fuzzy set (IFS) that takes into account not only the degree of membership but also the degree of non-membership and the inherent hesitation. IFS is a strong tool in domains where ambiguity is inherent because it provides a more sophisticated representation of uncertainty than the typical fuzzy set, which just takes membership into consideration.

As an expansion of his previous idea of IFS, Atanassov [4] introduced the concept of intuitionistic fuzzy graph (IFG). A more accurate method for simulating uncertainty in network interactions is provided by an IFG. It allows for a better portrayal of ambiguity in real-world problems by incorporating non-membership and hesitation degrees in addition to the membership degree.

The structure of this document is as follows: To establish the foundation for the suggested paradigm, some fundamental concepts and findings are given in Section 2. The q-fractional hesitancy fuzzy graph models, along with their main functions and outcomes, are formally presented in Section 3.

2 Preliminaries

Definition 2.1.

Let $\mathcal{V}$ be a set that is non-empty. A fuzzy graph is defined as a pair of functions $\mathcal{G} = (\sigma, \zeta)$ where σ is a fuzzy subset of $\mathcal{V}$, ζ is a symmetric fuzzy relation on σ . that is, $\sigma: \mathcal{V} \to [0,1]$ and $\zeta: \mathcal{V} \times \mathcal{V} \to [0,1]$ such that $\zeta(u, v) \leq \sigma(u) \wedge \sigma(v)$ for all $u, v \in \mathcal{V}$. The underlying crisp graph of the fuzzy graph $\mathcal{G} = (\sigma, \zeta)$ is denoted as $\mathcal{G}^* = (\sigma^*, \zeta)$, where σ* is referred to as the nonempty set $\mathcal{V}$ of nodes and $\zeta = \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. $\wedge$ represents the minimum.

Definition 2.2.

A Hesitancy Fuzzy Graphs is of the form $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{u_1, u_2, \cdots u_n\}$ such that $\zeta_{\mathcal{A}}: \mathcal{V} \to [0,1]$, $\eta_{\mathcal{A}}: \mathcal{V} \to [0, \mathcal{A}]$ and $\tau_{\mathcal{A}}: \mathcal{V} \to [0,1]$ denote the degrees of membership, non-membership and hesitancy of the vertex $u_i \in \mathcal{V}$ respectively and $\zeta_{\mathcal{A}}(u_i) + \eta_{\mathcal{A}}(u_i) + \tau_{\mathcal{A}}(u_i) = 1$ for every $u_i \in \mathcal{V}$, where $\tau_{\mathcal{A}}(u_i) = 1 - [\zeta_{\mathcal{A}}(u_i) + \eta_{\mathcal{A}}(u_i)]$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ where $\zeta_{\mathcal{B}}: \mathcal{V} \times \mathcal{V} \to [0,1]$,$\eta_{\mathcal{B}}: \mathcal{V} \times \mathcal{V} \to [0,1]$ and$\tau_{\mathcal{B}}: \mathcal{V} \times \mathcal{V} \to [0,1]$ are such that, $\zeta_{\mathcal{B}}(u_i, u_j) = min[\zeta_{\mathcal{A}}(u_i), \zeta_{\mathcal{A}}(u_j)]$, $\eta_{\mathcal{B}}(u_i, u_j) \leq max[\eta_{\mathcal{A}}(u_i), \eta_{\mathcal{A}}(u_j)]$, $\tau_{\mathcal{B}}(u_i, u_j) \leq min[\tau_{\mathcal{A}}(u_i), \tau_{\mathcal{A}}(u_j)]$ and $0 \leq \zeta_{\mathcal{B}}(u_i, u_j) +$

$\eta_{\mathcal{B}}(u_i, u_j) + \tau_{\mathcal{B}}(u_i, u_j) \leq 1$ for every $(u_i, u_j) \in \mathcal{E}$.

Definition 2.3.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a Hesitancy fuzzy graph. The degree of a vertex u is $d(u) = (d\zeta_{\mathcal{A}}(u), d\eta_{\mathcal{A}}(u), d\tau_{\mathcal{A}}(u))$, where $d\zeta_{\mathcal{A}}(u) = \sum_{u \neq u} \zeta_{\mathcal{B}}(u, v)$, $d\eta_{\mathcal{A}}(u) = \sum_{u \neq u} \eta_{\mathcal{B}}(u, v)$ and $d\tau_{\mathcal{A}}(u) = \sum_{u \neq u} \tau_{\mathcal{B}}(u, v)$ for all $u, v \in \mathcal{V}$.

3 The q – fractional Hesitancy fuzzy graph

Definition 3.1.

A q – fraction fuzzy set over a non-empty set $\mathcal{X}$ is defined as $\mathcal{A} = \{x, \zeta_{\mathcal{A}}(x), \eta_{\mathcal{A}}(x) / x \in \mathcal{X}\}$ where $\zeta_{\mathcal{A}}(x), \eta_{\mathcal{A}}(x) \in [0,1]$ are degree of membership and degree of non- membership, that satisfy the constraint $0 \leq \frac{\zeta_{\mathcal{A}}(x)}{q} + \frac{\eta_{\mathcal{A}}(x)}{q} \leq 1$ when $q \geq 2$.

Definition 3.2.

A q – fraction Hesitancy fuzzy set over a non-empty universal set $\mathcal{X}$ is defined as follows $\mathcal{A} = \{x, \zeta_{\mathcal{A}}(x), \eta_{\mathcal{A}}(x), \tau_{\mathcal{A}}(x)\} / x \in \mathcal{X}\}$ where $\zeta_{\mathcal{A}}(x), \eta_{\mathcal{A}}(x), \tau_{\mathcal{A}}(x) \in [0,1]$ are degree of membership, degree of non-membership and degree of hesitant respectively, satisfying the condition $0 \leq \frac{\zeta_{\mathcal{A}}(x)}{q} + \frac{\eta_{\mathcal{A}}(x)}{q} + \frac{\tau_{\mathcal{A}}(x)}{q} \leq 1$ when $q \geq 2$.

Definition 3.3.

Let $\mathcal{A}_1 = \{\zeta_{\mathcal{A}_1}(x), \eta_{\mathcal{A}_1}(x), \tau_{\mathcal{A}_1}(x)\}$ and $\mathcal{A}_2 = \{\zeta_{\mathcal{A}2}(x), \eta_{\mathcal{A}2}(x), \tau_{\mathcal{A}2}(x)\}$ be two q – fraction hesitancy fuzzy set such that $0 \leq \frac{\zeta_{\mathcal{A}1}(x)}{q} + \frac{\eta_{\mathcal{A}1}(x)}{q} + \frac{\tau_{\mathcal{A}1}(x)}{q} \leq 1$ and $0 \leq \frac{\zeta_{\mathcal{A}2}(x)}{q} + \frac{\eta_{\mathcal{A}2}(x)}{q} + \frac{\tau_{\mathcal{A}2}(x)}{q} \leq 1$, when $q \geq 2$, then the following operations are defined by

(i) $\mathcal{A}_{1 \subseteq_P} \mathcal{A}_2$ if $\zeta_{\mathcal{A}_1}(x) \leq \zeta_{\mathcal{A}2}(x), \eta_{\mathcal{A}_1}(x) \leq \eta_{\mathcal{A}2}(x)$ and $\tau_{\mathcal{A}_1}(x) \leq \tau_{\mathcal{A}2}(x)$

(ii) $\mathcal{A}_{1 \subseteq_R} \mathcal{A}_2$ if $\zeta_{\mathcal{A}_1}(x) \leq \zeta_{\mathcal{A}2}(x), \eta_{\mathcal{A}_1}(x) \geq \eta_{\mathcal{A}2}(x)$ and $\tau_{\mathcal{A}_1}(x) \leq \tau_{\mathcal{A}2}(x)$

(iii) $\mathcal{A}_1 = \mathcal{A}_2$ if $\zeta_{\mathcal{A}_1}(x) = \zeta_{\mathcal{A}2}(x), \eta_{\mathcal{A}_1}(x) = \eta_{\mathcal{A}2}(x)$ and $\tau_{\mathcal{A}_1}(x) = \tau_{\mathcal{A}2}(x)$

(iv) $\mathcal{A}_1 \cup_P \mathcal{A}_2 = \{\max\{\zeta_{\mathcal{A}_1}(x), \zeta_{\mathcal{A}2}(x), \max\{\eta_{\mathcal{A}_1}(x), \eta_{\mathcal{A}2}(x)\}, \max\{\tau_{\mathcal{A}_1}(x), \tau_{\mathcal{A}2}(x)\}\}$

(v) $\mathcal{A}_1 \cup_R \mathcal{A}_2 = \{\min\{\zeta_{\mathcal{A}_1}(x), \zeta_{\mathcal{A}2}(x), \max\{\eta_{\mathcal{A}_1}(x), \eta_{\mathcal{A}2}(x)\}, \min\{\tau_{\mathcal{A}_1}(x), \tau_{\mathcal{A}2}(x)\}\}$

(vi) $\mathcal{A}_1 \cap_P \mathcal{A}_2 = \{\min\{\zeta_{\mathcal{A}_1}(x), \zeta_{\mathcal{A}2}(x), \min\{\eta_{\mathcal{A}_1}(x), \eta_{\mathcal{A}2}(x)\}, \min\{\tau_{\mathcal{A}_1}(x), \tau_{\mathcal{A}2}(x)\}\}$

(vii) $\mathcal{A}_1 \cap_R \mathcal{A}_2 = \{\min\{\zeta_{\mathcal{A}_1}(x), \zeta_{\mathcal{A}2}(x), \max\{\eta_{\mathcal{A}_1}(x), \eta_{\mathcal{A}2}(x)\}, \min\{\tau_{\mathcal{A}_1}(x), \tau_{\mathcal{A}2}(x)\}\}$

(viii) $\mathcal{A}^C = \{1 - \zeta_{\mathcal{A}}(x), 1 - \eta_{\mathcal{A}}(x), 1 - \tau_{\mathcal{A}}(x)\}$

Proposition 3.4.

Let $\mathcal{A}_1 = (\zeta_{\mathcal{A}_1}(x), \eta_{\mathcal{A}_1}(x), \tau_{\mathcal{A}_1}(x))$ such that $0 \leq \frac{\zeta_{\mathcal{A}_1}(x)}{q} + \frac{\eta_{\mathcal{A}_1}(x)}{q} + \frac{\tau_{\mathcal{A}_1}(x)}{q} \leq 1$ and $\mathcal{A}_2 = (\zeta_{\mathcal{A}2}(x), \eta_{\mathcal{A}2}(x), \tau_{\mathcal{A}2}(x))$ such that $0 \leq \frac{\zeta_{\mathcal{A}2}(x)}{q} + \frac{\eta_{\mathcal{A}2}(x)}{q} + \frac{\tau_{\mathcal{A}2}(x)}{q} \leq 1$ and $\mathcal{A}_3 = (\zeta_{\mathcal{A}_3}(x), \eta_{\mathcal{A}_3}(x), \tau_{\mathcal{A}_3}(x))$ such that $0 \leq \frac{\zeta_{\mathcal{A}_3}(x)}{q} + \frac{\eta_{\mathcal{A}_3}(x)}{q} + \frac{\tau_{\mathcal{A}_3}(x)}{q} \leq 1$ when $q \geq 2$ be any three q – fractional Hesitancy fuzzy sets. Then

(i) If $\mathcal{A}_1 \subseteq_P \mathcal{A}_2 \subseteq_P \mathcal{A}_3$ then $\mathcal{A}_1 \subseteq_P \mathcal{A}_3$

(ii) If $\mathcal{A}_1 \subseteq_R \mathcal{A}_2 \subseteq_R \mathcal{A}_3$ then $\mathcal{A}_1 \subseteq_R \mathcal{A}_3$

(iii) $(\mathcal{A}^C)^C = \mathcal{A}$

(iv) $(\mathcal{A}_1 \cup_P \mathcal{A}_2)^C = \mathcal{A}_1^C \cap_P \mathcal{A}_2^C$

(v) $(\mathcal{A}_1 \cup_R \mathcal{A}_2)^C = \mathcal{A}_1^C \cap_R \mathcal{A}_2^C$

(vi) $(\mathcal{A}_1 \cap_P \mathcal{A}_2)^C = \mathcal{A}_1^C \cup_P \mathcal{A}_2^C$

(vii) $(\mathcal{A}_1 \cap_R \mathcal{A}_2)^C = \mathcal{A}_1^C \cup_R \mathcal{A}_2^C$

Definition 3.5.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a simple graph with $\mathcal{V}$ as a non-empty set of vertices and $\mathcal{E} \subseteq \mathcal{V} X \mathcal{V}$ as a set of edges. A q – fractional Hesitancy fuzzy graph is defined as $\mathcal{G}_{qH} = (\mathcal{A}, \mathcal{B})$ where $\mathcal{A} = \{u_i, \zeta_{\mathcal{A}}(u_i), \eta_{\mathcal{A}}(u_i), \tau_{\mathcal{A}}(u_i) / u_i \in \mathcal{V}\}$ is a q – fractional Hesitancy fuzzy vertex set. $\mathcal{B} = \{(u_i, u_j), \zeta_{\mathcal{B}}(u_i, u_j), \eta_{\mathcal{B}}(u_i, u_j), \tau_{\mathcal{B}}(u_i, u_j) / ((u_i, u_j) \in \mathcal{E}\}$ ia a Hesitancy fuzzy q – fractional edge set Such that the following conditions are satisfied

(i) For vertex degree of membership, non-membership and hesitant mappings $\zeta_{\mathcal{A}}, \eta_{\mathcal{A}}, \tau_{\mathcal{A}} : \mathcal{V} \to [0,1]$ such that $\zeta_{\mathcal{A}}(u_i) + \eta_{\mathcal{A}}(u_i) + \tau_{\mathcal{A}}(u_i) = 1$ and $0 \leq \frac{\zeta_{\mathcal{A}}(u_i)}{q} + \frac{\eta_{\mathcal{A}}(u_i)}{q} + \frac{\tau_{\mathcal{A}}(u_i)}{q} \leq 1$ for all $u_i \in \mathcal{V}$; $i = 1,2, \ldots n$ with $q \geq 2$

(ii) For edge degree of membership, non-membership and hesitancy mappings $\zeta_{\mathcal{B}}, \eta_{\mathcal{B}}, \tau_{\mathcal{B}} : \mathcal{V} X \mathcal{V} \to [0,1]$ such that $0 \leq \frac{\zeta_{\mathcal{B}}(u_i, u_j)}{q} + \frac{\eta_{\mathcal{B}}(u_i, u_j)}{q} + \frac{\tau_{\mathcal{B}}(u_i, u_j)}{q} \leq 1$ $0 \leq \zeta_{\mathcal{B}}(u_i, u_j) + \eta_{\mathcal{B}}(u_i, u_j) + \tau_{\mathcal{B}}(u_i, u_j) \leq 1$; $i, j = 1,2,3, \ldots n$ for all $(u_i, u_j) \in \mathcal{E}$ with $q \geq 2$ also $\zeta_{\mathcal{B}}(u_i, u_j) \leq \min\{\zeta_{\mathcal{A}}(u_i), \zeta_{\mathcal{A}}(u_j)\}$, $\eta_{\mathcal{B}}(u_i, u_j) \leq \max\{\eta_{\mathcal{A}}(u_i), \eta_{\mathcal{A}}(u_j)\}$, $\tau_{\mathcal{B}}(u_i, u_j) \leq \min\{\tau_{\mathcal{A}}(u_i), \tau_{\mathcal{A}}(u_j)\}$

Example 3.6.

Consider the q – fraction Hesitancy fuzzy graph with $\mathcal{V} = (x, y, z, w)$, $\mathcal{E} = \{xy, xz, yz, zw\}$

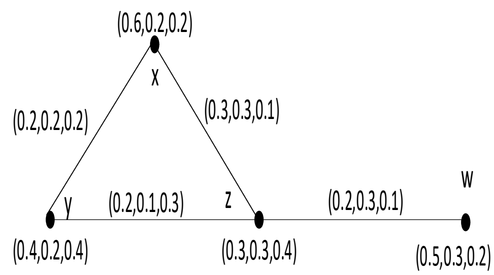


Figure -1 : q – fractional Hesitancy fuzzy graph

Consider $q = 2$ For x, $\zeta_{\mathcal{A}}(x) = 0.6$, $\eta_{\mathcal{A}}(x) = 0.2$, $\tau_{\mathcal{A}}(x) = 0.2$ $\zeta_{\mathcal{A}}(x) + \eta_{\mathcal{A}}(x) + \tau_{\mathcal{A}}(x) = 1$ and $\frac{\zeta_{\mathcal{A}}(x)}{2} = \frac{0.6}{2} = 0.3$, $\frac{\eta_{\mathcal{A}}(x)}{2} = \frac{0.2}{2} = 0.1$, $\frac{\tau_{\mathcal{A}}(x)}{2} = \frac{0.2}{2} = 0.1$, $\frac{\zeta_{\mathcal{A}}(x)}{q} + \frac{\eta_{\mathcal{A}}(x)}{q} + \frac{\tau_{\mathcal{A}}(x)}{q} = 0.3 + 0.1 + 0.1 = 0.5$

Each vertex has degrees of membership, non-membership and hesitancy that satisfy the q – fractional constraint.

$$0 \leq \frac{\zeta_{\mathcal{A}}(u_i)}{q} + \frac{\eta_{\mathcal{A}}(u_i)}{q} + \frac{\tau_{\mathcal{A}}(u_i)}{q} \leq 1.$$

Similarly , each edge has membership, non-membership and hesitancy degrees that satisfies the q – fractional constraint.

$0 \leq \frac{\zeta_{\mathcal{B}}(u_i, u_j)}{q} + \frac{\eta_{\mathcal{B}}(u_i, u_j)}{q} + \frac{\tau_{\mathcal{B}}(u_i, u_j)}{q} \leq 1.$

Definition 3.7.

The order of a q – fractional Hesitancy fuzzy graph $\mathcal{G}_{qH} = (\mathcal{A}, \mathcal{B})$ is defined as $O(\mathcal{G}_{qH}) = (\sum_{u_i \in \mathcal{V}} \zeta_{\mathcal{A}}(u_i), \sum_{u_i \in \mathcal{V}} \eta_{\mathcal{A}}(u_i), \sum_{u_i \in \mathcal{V}} \tau_{\mathcal{A}}(u_i))$

Example 3.8.

Consider the q – fractional Hesitancy fuzzy graph $\mathcal{G}_{qH} = (\mathcal{A}, \mathcal{B})$ with $\mathcal{V} = \{v_1, v_2, v_3, v_4\}$ and $\mathcal{E} = \{v_1 v_2, v_1 v_3, v_3 v_4, v_2 v_4\}$

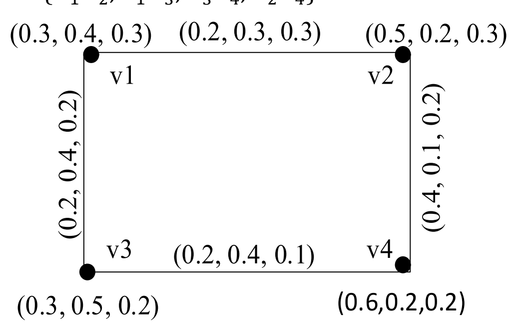


Figure – 2 : q – fractional Hesitancy fuzzy graph

Order of $\mathcal{G}_{qH} = (1.7, 1.3, 1.0) = (0.3 + 0.5 + 0.3 + 0.6,\ 0.4 + 0.2 + 0.5 + 0.2,\ 0.3 + 0.3 + 0.2 + 0.2)$

Definition 3.9.

The degree of a vertex $u_i \in \mathcal{V}$ in a q – fractional Hesitancy fuzzy graph $\mathcal{G}_{qH} = (\mathcal{A}, \mathcal{B})$ is a defined as $\deg(u_i) = (\deg \zeta_{\mathcal{A}}(u_i), \deg \eta_{\mathcal{A}}(u_i), \deg \tau_{\mathcal{A}}(u_i))$, where $\deg \zeta_{\mathcal{A}}(u_i) = \sum_{u_i \neq u_j} \zeta_{\mathcal{B}}(u_i, u_j)$,

$\deg \eta_{\mathcal{A}}(u_i) = \sum_{u_i \neq u_j} \eta_{\mathcal{B}}(u_i, u_j)$ and

$\deg \tau_{\mathcal{A}}(u_i) = \sum_{u_i \neq u_j} \tau_{\mathcal{B}}(u_i, u_j)$, where $0 \leq \frac{\zeta_{\mathcal{B}}(u_i, u_j)}{q} + \frac{\eta_{\mathcal{B}}(u_i, u_j)}{q} + \frac{\tau_{\mathcal{B}}(u_i, u_j)}{q} \leq 1.$

Example 3.10.

Consider the q – fractional Hesitancy fuzzy graph $\mathcal{G}_{qH} = (\mathcal{A}, \mathcal{B})$ with $\mathcal{V} = \{u, v, w, x\}$ and $\mathcal{E} = \{uv, uw, vw, wx\}$

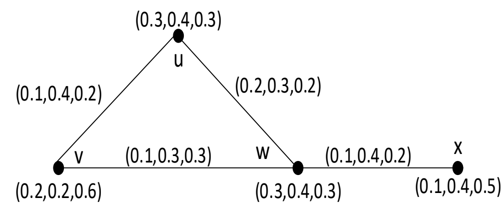


Figure – 3 : q – fractional Hesitancy fuzzy graph

Here $d(u) = (0.2 + 0.3, 0.3 + 0.3, 0.2 + 0.2) = (0.5, 0.6, 0.4)$, $d(v) = (0.2 + 0.3, 0.3 + 0.4, 0.2 + 0.2) = (0.5, 0.7, 0.4)$, $d(w) = (0.3 + 0.3 + 0.3, 0.3 + 0.4 + 0.3, 0.2 + 0.2 + 0.2) = (0.9, 1.0, 0.6)$ and $d(x) = (0.3, 0.3, 0.2)$.

4 Operations on q – fractional Hesitancy fuzzy graph

Definition 4.1.

Let $\mathcal{G}_{qH1} = (\mathcal{A}_1, \mathcal{B}_1)$ and $\mathcal{G}_{qH2} = (\mathcal{A}_2, \mathcal{B}_2)$ be two q – fractional fuzzy graphs then their cartesian product is defined as $\mathcal{G}_{qH1} \times \mathcal{G}_{qH2} = (\mathcal{A}_1 \times \mathcal{A}_2, \mathcal{B}_1 \times \mathcal{B}_2)$ such that the following conditions are satisfied

(i) For all $v_1, v_2 \in \mathcal{V}$, $\zeta_{\mathcal{A}_1 \times \mathcal{A}_2}(v_1, v_2) = \min\{\zeta_{\mathcal{A}_1}(v_1), \zeta_{\mathcal{A}_2}(v_2)\}$, $\eta_{\mathcal{A}_1 \times \mathcal{A}_2}(v_1, v_2) = \max\{\eta_{\mathcal{A}_1}(v_1), \eta_{\mathcal{A}_2}(v_2)\}$ and $\tau_{\mathcal{A}_1 \times \mathcal{A}_2}(v_1, v_2) = \min\{\tau_{\mathcal{A}_1}(v_1), \tau_{\mathcal{A}_2}(v_2)\}$.

(ii) for all $v \in \mathcal{V}_1$, $v_2 u_2 \in \mathcal{E}_2$, $\zeta_{\mathcal{B}_1 \times \mathcal{B}_2}((v, v_2), (v, u_2)) = \min\{\zeta_{\mathcal{A}_1}(v), \zeta_{\mathcal{B}_2}(v_2, u_2)$, $\eta_{\mathcal{B}_1 \times \mathcal{B}_2}((v, v_2), (v, u_2)) = \max\{\eta_{\mathcal{A}_1}(v), \eta_{\mathcal{B}_2}(v_2, u_2)\}$ and $\tau_{\mathcal{B}_1 \times \mathcal{B}_2}((v, v_2), (v, u_2)) = \min\{\tau_{\mathcal{A}_1}(v), \tau_{\mathcal{B}_2}(v_2, u_2)\}$.

(iii) For all $v \in \mathcal{V}_2$, $(v_1, u_1) \in \mathcal{E}_1$, $\zeta_{\mathcal{B}_1 \times \mathcal{B}_2}((v_1, v), (u_1, v)) = \min\{\zeta_{\mathcal{B}_1}(v_1, u_1), \zeta_{\mathcal{A}_2}(v)\}$, $\eta_{\mathcal{B}_1 \times \mathcal{B}_2}((v_1, v), (u_1, v)) = \max\{\eta_{\mathcal{B}_1}(v_1, u_1), \eta_{\mathcal{A}_2}(v)\}$ and $\tau_{\mathcal{B}_1 \times \mathcal{B}_2}((v_1, v), (u_1, v)) = \min\{\tau_{\mathcal{B}_1}(v_1, u_1), \tau_{\mathcal{A}_2}(v)\}$.

Proposition 4.2.

The cartesian product of two q – fractional Hesitancy graphs is a q – fractional Hesitancy fuzzy graph.

Definition 4.3.

Let $\mathcal{G}_{qH1} = (\mathcal{A}_1, \mathcal{B}_1)$ and $\mathcal{G}_{qH2} = (\mathcal{A}_2, \mathcal{B}_2)$ be two q – fractional Hesitancy fuzzy graphs, then their P – union is defined as $\mathcal{G}_{qH1} \cup_P \mathcal{G}_{qH2} = (\mathcal{A}_1 \cup_P \mathcal{A}_2, \mathcal{B}_1 \cup_P \mathcal{B}_2)$ such that the following conditions are satisfied

(a) for all $u \in \mathcal{V}_1$, and $u \notin \mathcal{V}_2$
$\zeta_{\mathcal{A}_1 \cup_P \mathcal{A}_2}(u) = \zeta_{\mathcal{A}_1}(u)$,
$\eta_{\mathcal{A}_1 \cup_P \mathcal{A}_2}(u) = \eta_{\mathcal{A}_1}(u)$ and
$\tau_{\mathcal{A}_1 \cup_P \mathcal{A}_2}(u) = \tau_{\mathcal{A}_1}(u)$.

(b) for all $u \in \mathcal{V}_2$, and $u \notin \mathcal{V}_1$
$\zeta_{\mathcal{A}_1 \cup_P \mathcal{A}_2}(u) = \zeta_{\mathcal{A}_2}(u)$,
$\eta_{\mathcal{A}_1 \cup_P \mathcal{A}_2}(u) = \eta_{\mathcal{A}_2}(u)$ and
$\tau_{\mathcal{A}_1 \cup_P \mathcal{A}_2}(u) = \tau_{\mathcal{A}_2}(u)$.

(c) for all $u \in \mathcal{V}_1 \cap \mathcal{V}_2$
$\zeta_{\mathcal{A}_1 \cup_P \mathcal{A}_2}(u) = \max\{\zeta_{\mathcal{A}_1}(u), \zeta_{\mathcal{A}_2}(u)\}$,
$\eta_{\mathcal{A}_1 \cup_P \mathcal{A}_2}(u) = \max\{\eta_{\mathcal{A}_1}(u), \eta_{\mathcal{A}_2}(u)\}$ and
$\tau_{\mathcal{A}_1 \cup_P \mathcal{A}_2}(u) = \max\{\tau_{\mathcal{A}_1}(u), \tau_{\mathcal{A}_2}(u)\}$.

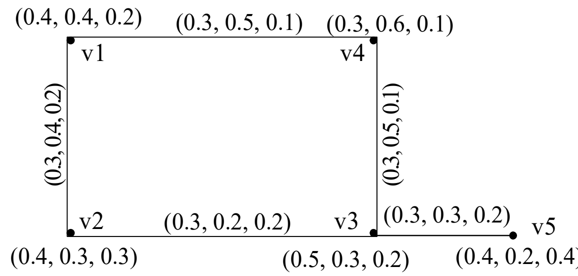
(d) for all $(u, w) \in \mathcal{E}_1$ and $(u, w) \notin \mathcal{E}_2$
$\zeta_{\mathcal{B}_1 \cup_P \mathcal{B}_2}(u, w) = \zeta_{\mathcal{B}_1}(u, w)$,
$\eta_{\mathcal{B}_1 \cup_P \mathcal{B}_2}(u, w) = \eta_{\mathcal{B}_1}(u, w)$ and
$\tau_{\mathcal{A}_1 \cup_P \mathcal{A}_2}(u) = 1 - \{\zeta_{\mathcal{A}_1 \cup_P \mathcal{A}_2}(u) + \eta_{\mathcal{A}_1 \cup_P \mathcal{A}_2}(u)\}$.

(e) for all $(u, v) \in \mathcal{E}_2$ and $(u, v) \notin \mathcal{E}_1$
$\zeta_{\mathcal{B}_1 \cup_P \mathcal{B}_2}(u, v) = \zeta_{\mathcal{B}_2}(u, v)$,
$\eta_{\mathcal{B}_1 \cup_P \mathcal{B}_2}(u, v) = \eta_{\mathcal{B}_2}(u, v)$ and
$\tau_{\mathcal{B}_1 \cup_P \mathcal{B}_2}(u, v) = \tau_{\mathcal{B}_2}(u, v)$.

(f) for all $(u, v) \in \mathcal{E}_1 \cap \mathcal{E}_2$
$\zeta_{\mathcal{B}_1 \cup_P \mathcal{B}_2}(u, v) = \max\{\zeta_{\mathcal{B}_1}(u, v), \zeta_{\mathcal{B}_2}(u, v)\}$,
$\eta_{\mathcal{B}_1 \cup_P \mathcal{B}_2}(u, v) = \max\{\eta_{\mathcal{B}_1}(u, v), \eta_{\mathcal{B}_2}(u, v)\}$ and
$\tau_{\mathcal{B}_1 \cup_P \mathcal{B}_2}(u, v) = \max\{\tau_{\mathcal{B}_1}(u, v), \tau_{\mathcal{B}_2}(u, v)\}$.

Example 4.4.

Consider the q – fractional Hesitancy fuzzy graphs $\mathcal{G}_{qH}$ and $\mathcal{G}_{qH}$



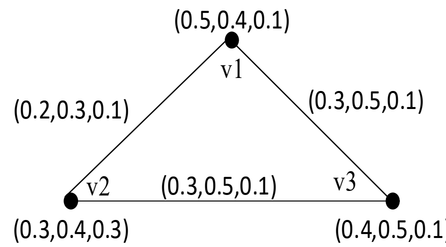


Figure -4: $\mathcal{G}_{qH1}$

Figure -5: $\mathcal{G}_{qH2}$

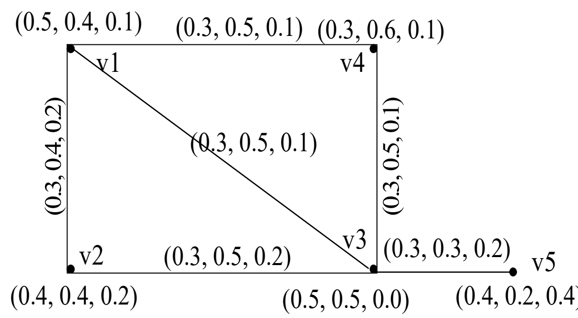


Figure – 6: $\mathcal{G}_{qH1} \cup_P \mathcal{G}_{qH2}$

Proposition 4.5.

The P – union of two q – fractional Hesitancy fuzzy graph is an q – fractional Hesitancy fuzzy graph.

Definition 4.6.

Let $\mathcal{G}_{qH1} = (\mathcal{A}_1, \mathcal{B}_1)$ and $\mathcal{G}_{qH} = (\mathcal{A}_2, \mathcal{B}_2)$ be two q – fractional Hesitancy fuzzy graphs, then their R – union is defined as follows $\mathcal{G}_{qH} \cup_R \mathcal{G}_{qH2} = (\mathcal{A}_1 \cup_R \mathcal{A}_2, \mathcal{B}_1 \cup_R \mathcal{B}_2)$ where $\mathcal{A}_1 \cup_R \mathcal{A}_2$ satisfied the following conditions.

(i) for all $v \in \mathcal{V}_1$ and $v \notin \mathcal{V}_2$,
$\zeta_{\mathcal{A}_1 \cup_R \mathcal{A}_2}(v) = \zeta_{\mathcal{A}_1}(v)$,
$\eta_{\mathcal{A}_1 \cup_R \mathcal{A}_2}(v) = \eta_{\mathcal{A}_1}(v)$ and
$\tau_{\mathcal{A}_1 \cup_R \mathcal{A}_2}(v) = \tau_{\mathcal{A}_1}(v)$.

(ii) for all $v \in \mathcal{V}_2$, and $v \notin \mathcal{V}_1$,
$\zeta_{\mathcal{A}_1 \cup_R \mathcal{A}_2}(v) = \zeta_{\mathcal{A}_2}(v)$,
$\eta_{\mathcal{A}_1 \cup_R \mathcal{A}_2}(v) = \eta_{\mathcal{A}_2}(v)$ and
$\tau_{\mathcal{A}_1 \cup_R \mathcal{A}_2}(v) = \tau_{\mathcal{A}_2}(v)$.

(iii) for all $v \in \mathcal{V}_1 \cap \mathcal{V}_2$,
$\zeta_{\mathcal{A}_1 \cup_R \mathcal{A}_2}(v) = \max\{\zeta_{\mathcal{A}_1}(v), \zeta_{\mathcal{A}_2}(v)\}$,
$\eta_{\mathcal{A}_1 \cup_R \mathcal{A}_2}(v) = \min\{\eta_{\mathcal{A}_1}(v), \eta_{\mathcal{A}_2}(v)\}$ and

$\tau_{\mathcal{A}_1 \cup_R \mathcal{A}_2}(v) = 1 - \{ \zeta_{\mathcal{A}_1 \cup_R \mathcal{A}_2}(v) + \eta_{\mathcal{A}_1 \cup_R \mathcal{A}_2}(v)\}$

and $\mathcal{B}_1 \cup_R \mathcal{B}_2$ satisfies the following conditions.

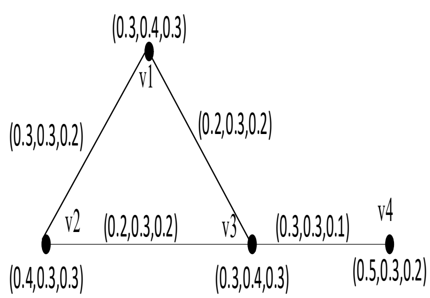
(i) for all $(u, v) \in \mathcal{E}_1$ and $(u, v) \notin \mathcal{E}_2$,
$\zeta_{\mathcal{B}_1 \cup_R \mathcal{B}_2}(u, v) = \zeta_{\mathcal{B}_1}(u, v)$,
$\eta_{\mathcal{B}_1 \cup_R \mathcal{B}_2}(u, v) = \eta_{\mathcal{B}_1}(u, v)$ and
$\tau_{\mathcal{B}_1 \cup_R \mathcal{B}_2}(u, v) = \tau_{\mathcal{B}_1}(u, v)$.

(ii) for all $(u, v) \in \mathcal{E}_2$ and $(u, v) \notin \mathcal{E}_1$,
$\zeta_{\mathcal{B}_1 \cup_R \mathcal{B}_2}(u, v) = \zeta_{\mathcal{B}_2}(u, v)$,
$\eta_{\mathcal{B}_1 \cup_R \mathcal{B}_2}(u, v) = \eta_{\mathcal{B}_2}(u, v)$ and
$\tau_{\mathcal{B}_1 \cup_R \mathcal{B}_2}(u, v) = \tau_{\mathcal{B}_2}(u, v)$.

(iii) for all $(u, v) = \mathcal{E}_1 \cap \mathcal{E}_2$,
$\zeta_{\mathcal{B}_1 \cup_R \mathcal{B}_2}(u, v) = \max\{ \zeta_{\mathcal{B}_1}(u, v), \zeta_{\mathcal{B}_2}(u, v)\}$,
$\eta_{\mathcal{B}_1 \cup_R \mathcal{B}_2}(u, v) = \min\{\eta_{\mathcal{B}_1}(u, v), \eta_{\mathcal{B}_2}(u, v)\}$ and
$\tau_{\mathcal{B}_1 \cup_R \mathcal{B}_2}(u, v) = \max\{ \tau_{\mathcal{B}_1}(u, v), \tau_{\mathcal{B}_2}(u, v)\}$.

Example 4.7.

Consider the q – fractional Hesitancy fuzzy graphs $\mathcal{G}_{qH1}$ and $\mathcal{G}_{qH}$



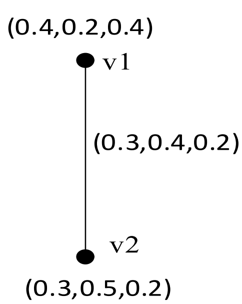


Figure -7: $\mathcal{G}_{qH}$

Figure -8: $\mathcal{G}_{qH2}$

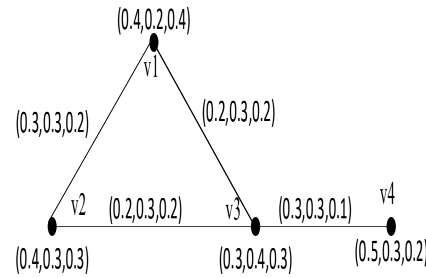


Figure – 6: $\mathcal{G}_{qH1} \cup_R \mathcal{G}_{qH2}$

Proposition 4.7.

A R – union q – fraction Hesitancy fuzzy graphs is a q – fraction Hesitancy fuzzy graph.

Conclusion

In this frame work introduced and formalized the framework of the q-fractional hesitancy fuzzy graph (q-FHFG) as a generalization of q-ROFG. To establish the theoretical foundations of q-FHFGs, including structural properties supported by graphical comparisons with existing fuzzy graph frameworks.

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