ASESDC Labeling for cycle of join zero divisor graphs

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Abstract
Average Square Even Sum Divisor Cordial Labeling (ASESDCL) of a graph G with vertex set $V(G)$ is a bijection $f: V(G) \to \{2,4,\ldots\ldots,2|V(G)|\}$ such that for every edge uv, f^* is defined as follows, $f^*(uv) = \{1 \quad if\ f(u) + f(v) | \left(\frac{f(u)+f(v)}{2}\right)^2 \ 0 \qquad elsewhere$ with the condition that $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$ where $e_{f^*}(0)$ & $e_{f^*}(1)$ denote the number of edges labeled with 1 and not labeled with 1. If G admits ASESDCL then G is Average Square Even Sum Divisor Cordial Graph (ASESDCG). In this paper, we investigate the ASESDCL of the cycle of join zero divisor graphs (ZDG). we also provide some examples to illustrate our main result.

Keywords: Zero Divisor graph, cycle of join zero divisor graph, ASESDCL, ASESDCG.
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1. INTRODUCTION

The studies of labeling play important roles in graph theory. Given that G is a graph, a labeling can be thought of as an allocation of numbers to both edges and vertices that meet certain requirements. Zero divisor graphs, introduced in [2], are a class of graphs that have gained significant attention in recent years due to their rich structural properties. The cycle of join zero divisor graph obtained by taking the join of G with a cycle of length 2. Labeling a graph with integers is a fundamental concept in graph theory that has numerous applications in various fields. In particular [4,8,9,10,11] has been studied extensively in recent years. All graphs considered in this paper we introduced the concept of ASESDCL for cycle of join zero divisor graphs.

2. PRELIMINARIES

Definition: 2.1 Let R be a commutative ring (with 1) and let $Z(R)$ be its set of zero-divisors. We associate a simple graph $\Gamma(R)$ to R with vertices $Z(R)^* = Z(R) - \{0\}$, the set of nonzero zero-divisors of R and for distinct $x, y \in Z(R)^*$, the vertices $x\ and\ y$ are adjacent if and only if $x, y = 0$.Thus $\Gamma(R)$ is the empty graph if and only if R is an integral domain.

Definition: 2.2 The path union of graph G is the graph obatained by adding an edge between corresponding vertices of G_i to $G_{i+1}, 1 \leq i \leq n-1$ where $G_1, G_2, \ldots. G_n (n \geq 2)$ are n copies of G. It is denoted by $P(n, G)$.

Definition: 2.3 For a cycle C_n, each vertex of C_n is replaced by connected graphs $G_1, G_2, \ldots. G_n$ known as the cycle of graphs. We shall denote it by $C(G_1, G_2, \ldots. G_n)$. If we replace each vertex by a graph G, i.e., $G_1 = G = G_2 = \cdots. G_n$ such a cycle of a graph G is denoted by $C(n, G)$.

Note: The terms are interchangeable.

3. Main Result:

Definition: 3.1 Let $G = (V, E)$ be a graph with p vertices and q edges G is said to have average square even sum divisor cordial labeling if there is a bijection $f: V(G) \to \{2,4,\ldots\ldots,2|V(G)|\}$ such that for every edge uv, f^* is defined as follows, $f^*(uv) = \{1 \quad if\ f(u) + f(v) | \left(\frac{f(u)+f(v)}{2}\right)^2 \ 0 \qquad elsewhere$

With the condition that $|e_{f^*}(0) - e_{f^*}(1)| \le 1$ where $e_{f^*}(0)$ & $e_{f^*}(1)$ denote the number of edges labeled with 1 and not labeled with 1. If G admits ASESDCL then G is ASESDCG.

Theorem: 3.2 The cycle of join ZDG $C\left(m, \Gamma(z_{2p}) + \Gamma(z_4)\right)$ where $p \ge 2, m \ge 3$ is ASESDCG.

Proof: consider $C\left(m, \Gamma(z_{2p}) + \Gamma(z_4)\right)$ be a cycle of join zero divisor graph

$case\ i)\ p = 2, m \ge 3$

The vertex set $V(G) = \{t_{1,1}, t_{2,1}, t_{3,1}, \dots \dots, t_{m,1}, s_{1,1}, s_{2,1}, \dots \dots., s_{m,1}\}$, the edge set $\{t_{j,1}, s_{j,1}\ where\ 1 \le j \le m\} \cup \{t_{j,1}, t_{j+1,1} where\ 1 \le j \le m-1\} \cup \{t_{1,1}, t_{m,1}\}$.Therefore$|V(G)| = 2m\ and\ |E(G)| = 2m. The\ labeling\ f: V \to \{2,4, \dots \dots, 2|V(G)|\}\ defined\ f(t_{j,1}) = 2(2j-1)\ where\ 1 \le j \le m, f(s_{j,1}) = 4k\ where\ 1 \le j \le m$. The edge labeling is $f^*(t_{j,1}s_{j,1}) = 0\ where\ 1 \le j \le m,\ f^*(t_{j,1}t_{j+1,1}) = 1\ where\ 1 \le j \le m-1, f^*(t_{1,1}t_{m,1}) = 1$. Therefore $|e_{f^*}(0) - e_{f^*}(1)| = |\frac{2n}{2} - \frac{2n}{2}|$. Hence ASESDCG.

$case\ ii)\ p \ge 3, m \ge 3$

The vertex set $V(G) = \{t_{1,1}, t_{1,2}, \dots, t_{1,p}, t_{2,1}, t_{2,2}, \dots, t_{2,p}, \dots t_{m,1}, t_{m,2}, \dots, t_{m,p}, s_{1,1}, s_{2,1}, \dots., s_{m,1}\}$ the edge set $E(G) = \{t_{j,k}t_{j,p}, t_{j,k}s_{j,1}, t_{j,p}s_{j,1}, t_{1,p}t_{m,p}\ 1 \le j \le m, 1 \le k \le p-1\} \cup \{t_{j,p}t_{j+1,p}\ 1 \le j \le m-1\}$

Therefore $|V(G)| = m(p+1)\ and\ |E(G)| = 2mp$

Define $f: V(G) \to \{2,4, \dots \dots, 2|V(G)|\}$

$f(t_{1,p}) = 2, f(t_{2,p}) = 2p + 4, f(t_{3,p}) = 4p + 6, \dots \dots. f(t_{m,p}) = 2(m-1)p + 2m$

$f(t_{1,k}) = 2k + 2$

$f(t_{2,k}) = 2k + 2p + 4$

$f(t_{3,k}) = 2k + 4p + 6$

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$f(t_{m,k}) = 2k + (m-1)2p + 4\ where\ 1 \le k \le p-1$

$f(s_{1,1}) = 2p + 2$

$f(s_{2,1}) = 4p + 4$

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$f(s_{m,1}) = 2(mp + m)$

Then the edge labeling as follows

$f^*(t_{j,k}t_{j,p}) = \{0\ \ if\ k\ is\ odd\ 1 \le j \le m, 1 \le k \le p-1\ \ 1\ if\ k\ is\ even\ 1 \le j \le m, 1 \le k \le p-1$

$f^*(t_{j,k}s_{j,1}) = \{0\ \ if\ k\ is\ even\ 1 \le j \le m, 1 \le k \le p-1\ \ 1\ \ if\ \ k\ is\ odd\ 1 \le j \le m, 1 \le k \le p-1$

$f^*(t_{j,p}s_{j,1}) = 0, f^*(t_{j,p}, t_{j+1,p}) = 1\ where\ 1 \le j \le m-1,\ f^*(t_{1,p}, t_{m,p}) = 1$

Therefore $|e_{f^*}(0) - e_{f^*}(1)| = |mp - mp| \le 1$

Hence the graph is ASESDCG.

Example: 3.3 A ASESDCG for $G = C(5, \Gamma(z_{10}) + \Gamma(z_4))$ is given in figure 3.3 where $m = 5, p = 5$ the vertex set $V(G) = \{t_{1,1}, t_{1,2}, \dots, t_{1,5}, t_{2,1}, t_{2,2}, \dots, t_{2,5}, \dots t_{5,1}, t_{5,2}, \dots, t_{5,p}\} \cup \{s_{1,1}, s_{2,1}, \dots, s_{m,1}\}$and the edge set $E(G) = \{t_{j,k}t_{j,p}, t_{j,k}s_{j,1}, t_{j,p}s_{j,1}, t_{1,p}t_{m,p}\ 1 \leq j \leq 5, 1 \leq k \leq 4\} \cup \{t_{j,p}t_{j+1,p}\}$. Therefore $|V(G)| = 30$ *and* $|E(G)| = 50$. $|e_{f^*}(0) - e_{f^*}(1)| = |25 - 25| \leq 1$

Hence the graph $G = C(5, \Gamma(z_{10}) + \Gamma(z_4))$ is ASESDCG.

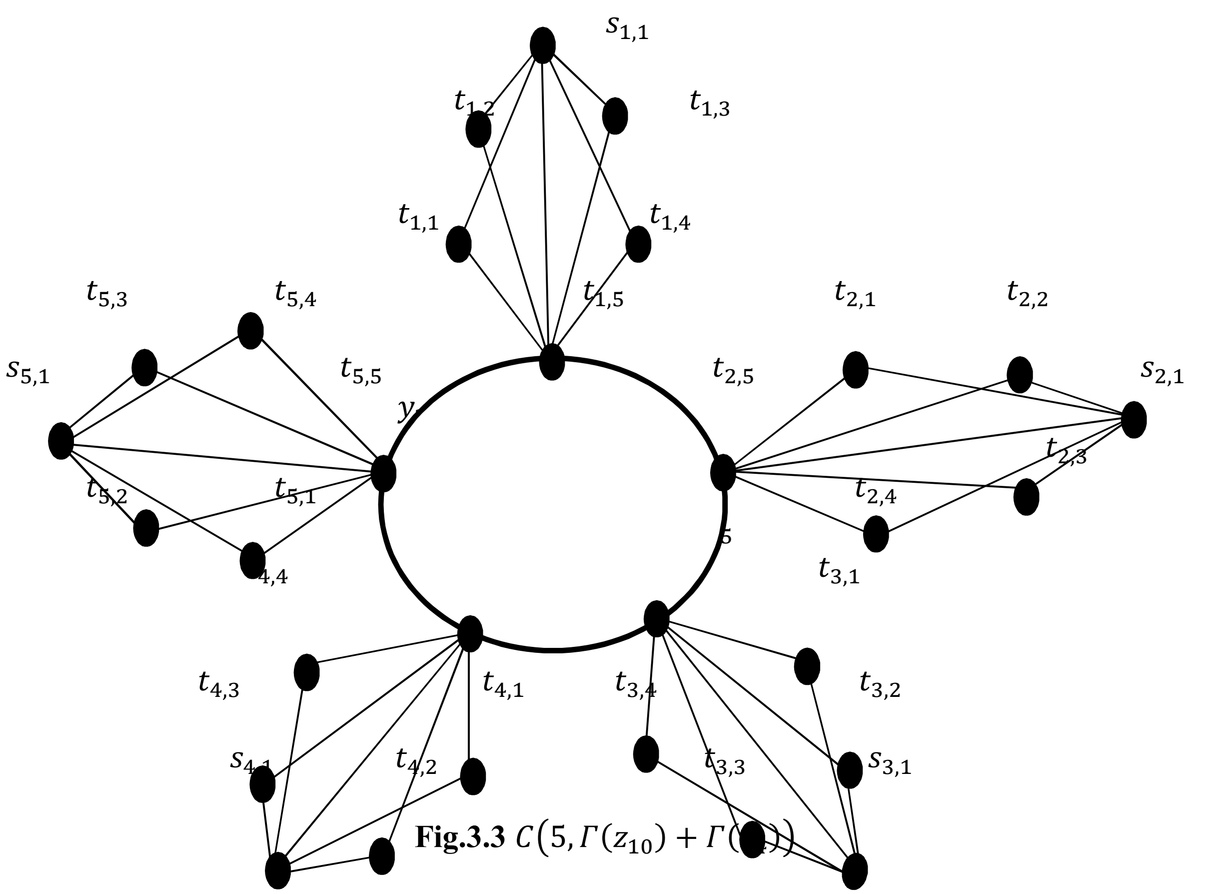

Fig.3.3 $C(5, \Gamma(z_{10}) + \Gamma(z_4))$

Theorem: 3.4 The cycle of join zero divisor graph $C\left(m, \Gamma(z_{2p}) + \Gamma(z_6)\right)$ where $p \geq 2, m \geq 3$ is ASESDCG.

Proof: consider $C\left(m, \Gamma(z_{2p}) + \Gamma(z_6)\right)$ be a cycle of join zero divisor graph

case i) $p = 2, m \geq 3$

The vertex set $V(G) = \{t_{1,1}, t_{2,1}, t_{3,1}, \dots \dots, t_{m,1}, s_{1,1}, s_{1,2}, s_{1,3}, s_{2,1}, s_{2,2}, s_{2,3}, \dots s_{m,1}, s_{m,2}, s_{m,3}\}$ and the edge set $E(G) = \{t_{j,1}s_{j,1}, t_{j,1}s_{j,2}, t_{j,1}s_{j,3}, s_{j,1}s_{j,2}, s_{j,2}s_{j,3}\ where\ 1 \leq j \leq m\} \cup \{t_{j,1}t_{j+1,1}, t_{1,1}t_{m,1}\ where\ 1 \leq j \leq m-1\}$

Therefore$|V(G)| = m(p+2)$ *and* $|E(G)| = m(p+4)$,*The labeling* $f: V(G) \to \{2,4, \dots \dots, 2|V(G)|\}$ *defined* $f(t_{j,1}) = p(4j-3)$, $f(s_{j,1}) = p(4j-3) + 2$, $f(s_{j,2}) =$

$p(4j-3)+6, f(s_{j,3}) = p(4j-3)+4$ where $1 \leq j \leq m$, The edge labeling is $f^*(t_{j,1}s_{j,1}) = 0, f^*(t_{j,1}s_{j,2}) = 0, f^*(t_{j,1}s_{j,3}) = 1, f^*(s_{j,1}s_{j,2}) = 1, f^*(s_{j,2}s_{j,3}) = 0, f^*(t_{j,1}t_{j+1,1}) = 1\ where\ 1 \leq j \leq m, f^*(t_{1,1}t_{m,1}) = 1$. Therefore $|e_{f^*}(0) - e_{f^*}(1)| = |\frac{(4+p)m}{2} - \frac{(4+p)m}{2}|$. The graph is ASESDCG.

$case\ ii)\ p \geq 3, m \geq 3$

The vertex set $V(G) = \{t_{1,1}, t_{1,2}, \ldots, t_{1,p}, t_{2,1}, t_{2,2}, \ldots, t_{2,p}, \ldots . t_{m,1}, t_{m,2}, \ldots, t_{m,p}, s_{1,1}, s_{1,2}, s_{1,3}, s_{2,1}, s_{2,2}, s_{2,3}, \ldots, s_{m,1}, s_{m,2}, s_{m,3}\}$

The edge set $E(G) = \{t_{j,k}t_{j,p}, t_{j,k}s_{j,1}, t_{j,k}s_{j,2}, t_{j,k}s_{j,3}, s_{j,1}s_{j,2}, s_{j,2}s_{j,3}, t_{j,p}s_{j,1}, t_{j,p}s_{j,2}, t_{j,p}s_{j,3}\ 1 \leq j \leq m, 1 \leq k \leq p-1\} \cup \{t_{j,p}\, t_{j+1,p}, t_{1,p}t_{m,p} 1 \leq j \leq m-1\}$

Therefore $|V(G)| = m(p+3)$ and $|E(G)| = 2m(2p+1)$

Define $f: V(G) \to \{2,4, \ldots \ldots, 2|V(G)|\}$

$f(t_{1,k}) = 2(k+1)$

$f(t_{2,k}) = 2(k+p+4)$

$f(t_{3,k}) = 2(k+2p+7)$

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$f(t_{m,k}) = 2(k+(m-1)p+(3m-2))$ where $1 \leq k \leq p-1$

$f(t_{1,p}) = 2$

$f(t_{2,p}) = 2(p+4)$

$f(t_{3,p}) = 2(2p+7)$

$f(t_{4,p}) = 2(3p+10)$

$f(t_{5,p}) = 2(4p+13)$

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$f(t_{m,p}) = 2((m-1)p+(3m-2))$

$f(s_{1,1}) = 2(p+1)$

$f(s_{2,1}) = 2(2p+4)$

$f(s_{3,1}) = 2(3p+7)$

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$f(s_{m,1}) = 2\,(mp+(3-2))$

$f(s_{1,2}) = 2(p+3)$

$f(s_{2,2}) = 2(2p+6)$

$f(s_{3,2}) = 2(3p+9)$

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$f(s_{m,2}) = 2\,(mp+3m)$

$f(s_{1,3}) = 2(p+2)$

$f(s_{2,3}) = 2(2p + 5)$
$f(s_{3,3}) = 2(3p + 8)$
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$f(s_{m,3}) = 2\ (mp + (3m - 1))$
Then the edge labeling as follows
$f^*(t_{j,k}t_{j,p}) = \{0\ \ if\ k\ is\ odd\ 1 \le j \le m, 1 \le k \le p-1\ \ 1\ if\ k\ is\ even\ 1 \le j \le m, 1 \le k \le p-1$
$f^*(t_{j,k}s_{j,1}) = \{0\ \ if\ k\ is\ even\ 1 \le j \le m, 1 \le k \le p-1\ \ 1\ \ if\ \ k\ is\ odd\ 1 \le j \le m, 1 \le k \le p-1$
$f^*(t_{j,k}s_{j,2}) = \{1\ \ \ if\ k\ is\ odd\ 1 \le j \le m, 1 \le k \le p-1\ \ 0\ \ if\ \ k\ is\ even\ 1 \le j \le m, 1 \le k \le p-1$
$f^*(t_{j,k}s_{j,3}) = \{1\ \ if\ k\ is\ even\ 1 \le j \le m, 1 \le k \le p-1\ \ 0\ \ \ if\ k\ is\ odd\ 1 \le j \le m, 1 \le k \le p-1$
$f^*(t_{j,p}s_{j,1}) = 0, f^*(t_{j,p}s_{j,3}) = 1, f^*(t_{j,p}s_{j,2}) = 0, f^*(s_{j,1}s_{j,2}) = 1, f^*(s_{j,2}s_{j,3}) = 0,$
$f^*(t_{j,p}u_{j+1,p}) = 1\ where\ 1 \le j \le m-1,\ f^*(t_{1,p}t_{m,p}) = 1$
Therefore $|e_{f^*}(0) - e_{f^*}(1)| = |m(2p+1) - m(2p+1)| \le 1$
Hence the graph is ASESDCG.

Example: 3.5 A ASESDCG for $G = C(4, \Gamma(z_{14}) + \Gamma(z_6))$ is given in figure 3.5 where $m = 4, p = 7$**.** The vertex set $V(G) = \{t_{1,1}, t_{1,2}, \dots, t_{1,7}, t_{2,1}, t_{2,2}, \dots, t_{2,7}, \dots t_{4,1}, t_{4,2}, \dots, t_{4,7}\} \cup \{s_{1,1}, s_{2,1}, s_{3,1}, s_{1,2}, s_{2,2}, s_{3,2}, s_{1,3}, s_{2,3}, s_{3,3}\}$The edge set $E(G) = \{t_{j,k}t_{j,7}, t_{j,k}s_{j,1}, t_{j,k}s_{j,2}, t_{j,k}s_{j,3}, t_{j,7}s_{j,1}, t_{j,7}s_{j,2}, t_{j,7}s_{j,3}, s_{j,1}\, s_{j,2}, s_{j,2}\, s_{j,3} 1 \le j \le 4, 1 \le k \le 6\} \cup \{t_{k,7}t_{k+1,7}, t_{1,7}t_{4,7}\ where\ 1 \le k \le 3\}$**.** Therefore $|\ V(G)| = 40\ and\ |E(G)\ | = 120$
$|e_{f^*}(0) - e_{f^*}(1)| = |60 - 60| \le 1$
Hence the graph $G =\ C(4, \Gamma(z_{14}) + \Gamma(z_6))$ is ASESDCG.

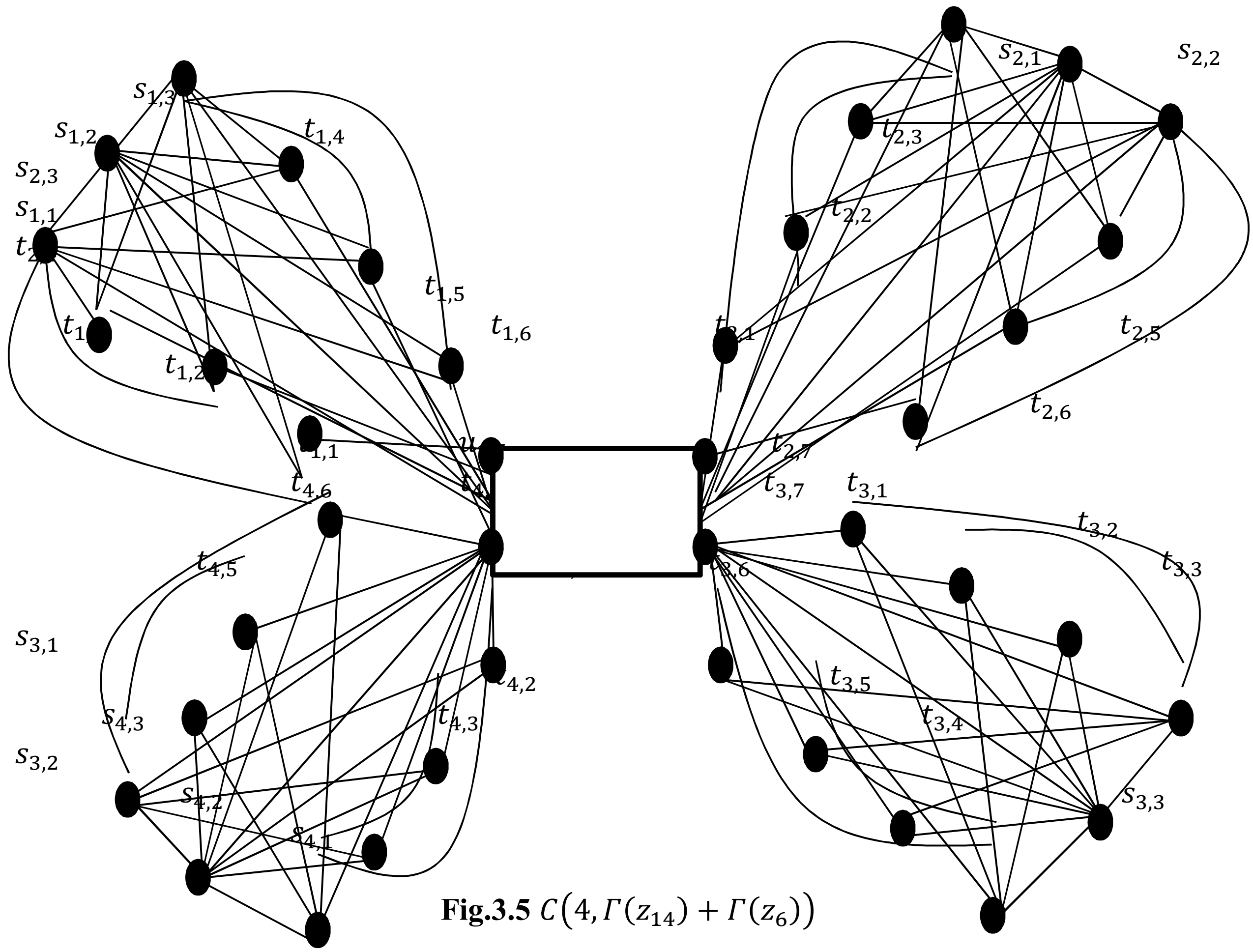

Fig.3.5 $C\left(4, \Gamma(z_{14}) + \Gamma(z_6)\right)$

Theorem: 3.6 The cycle of join zero divisor graph $C\left(m, \Gamma\left(z_{2p}\right) + \Gamma(z_9)\right)$ where $p \geq 3, m \geq 3$ does not ASESDCG.

Example: 3.7 A ASESDCG for $G = C\left(5, \Gamma(z_{10}) + \Gamma(z_9)\right)$ is given in figure 3.7 where $m = 5, p = 5$, the vertex set $V(G) = \{t_{1,1}, t_{1,2}, \dots, t_{1,5}, t_{2,1}, t_{2,2}, \dots, t_{2,5}, \dots t_{5,1}, t_{5,2}, \dots, t_{5,5}\} \cup \{s_{1,1}, s_{2,1}, \dots, s_{5,1}, s_{1,2}, s_{2,2}, \dots, s_{5,2}\}$, the edge set

$E(G) = \{t_{j,k}t_{j,5}, t_{j,k}s_{j,1}, t_{j,5}s_{j,1}, t_{j,k}s_{j,2}, t_{j,5}s_{j,2}, t_{1,5}\, t_{2,5}, t_{2,5}t_{3,5}, \dots t_{5,5}t_{1,5}\ 1 \leq j \leq 5, 1 \leq k \leq 4\}$

Therefore $|\,V(G)| = 35$ and $|E(G)\,| = 80$

$|e_{f^*}(0) - e_{f^*}(1)| = |44 - 36| = 8$

$\therefore |e_{f^*}(0) - e_{f^*}(1)| \nleq 1$

Hence the graph $G = C\left(5, \Gamma(z_{10}) + \Gamma(z_9)\right)$ does not admit ASESDCG.

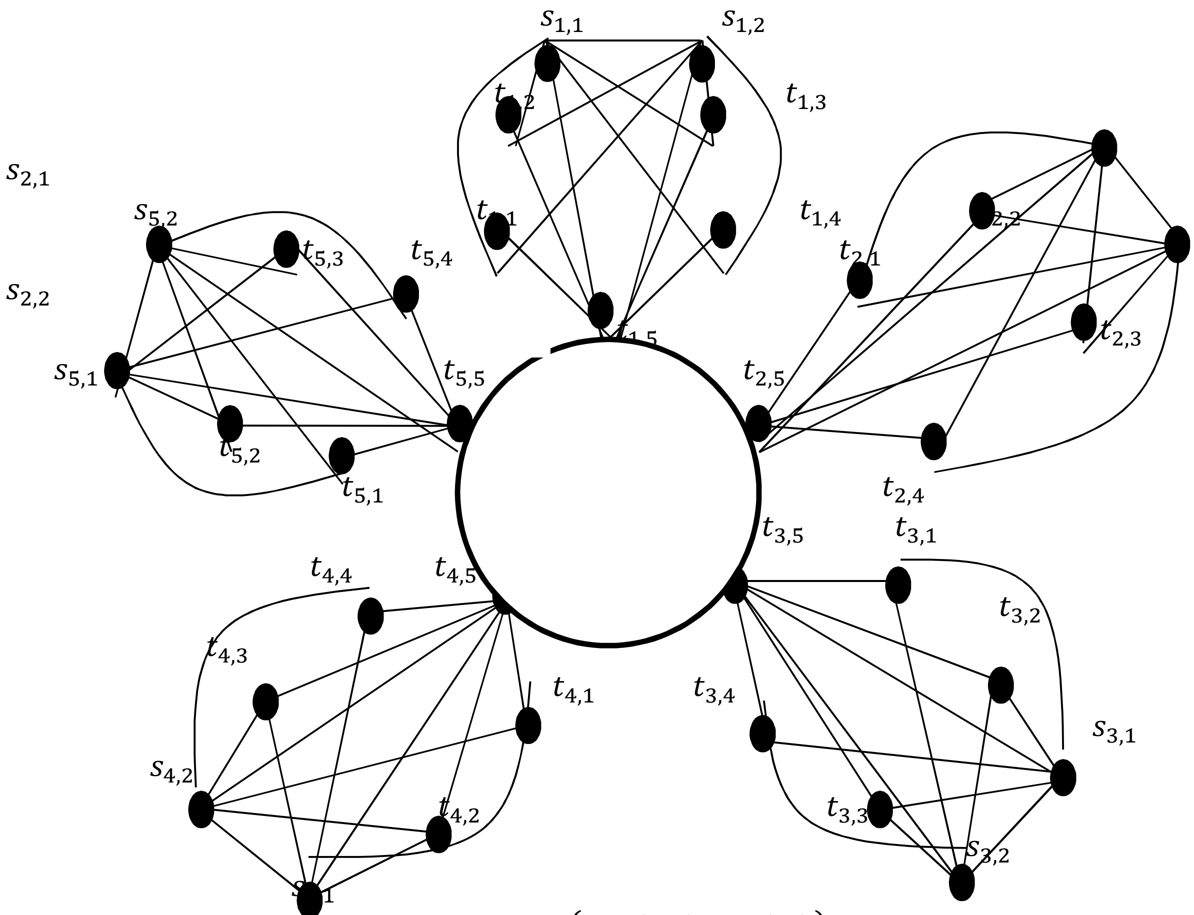

Fig. 3.7 $C(5, \Gamma(z_{10}) + \Gamma(z_9))$

Theorem: 3.8 The cycle of join zero divisor graph $C\left(3, \underline{\Gamma(z_{p^2})} + \Gamma(z_4)\right)$ where $p \geq 3, m = 3$ ASESDCG

Theorem: 3.9 The cycle of join zero divisor graph $C\left(3, \underline{\Gamma(z_{p^2})} + \Gamma(z_6)\right)$ where $p \geq 3, m = 3$ is ASESDCG.

Proof: $or\ p \geq 3, m = 3$, $C\left(3, \underline{\Gamma(z_{p^2})} + \Gamma(z_6)\right)$ be a cycle of join zero divisor graph, The vertex set $V(G) = \{t_{1,1}, t_{1,2}, \dots, t_{1,p-1}, t_{2,1}, t_{2,2}, \dots, t_{2,p-1}, \dots t_{m,1}, t_{m,2}, \dots, t_{m,p-1},\} \cup \{s_{1,1}, s_{1,2}, s_{1,3}, s_{2,1}, s_{2,2} s_{2,3}, \dots s_{m,1}, s_{m,2}, s_{m,3}\}$ and
the edge set $E(G) = \{t_{j,k}s_{j,1}, t_{j,k}s_{j,2}, t_{j,k}s_{j,3}, s_{j,1}s_{j,2}, s_{j,2}s_{j,3}\ 1 \leq j \leq m, 1 \leq k \leq p-1\} \cup \{t_{j,1}t_{j+1,1}, t_{1,1}t_{m,1} 1 \leq j \leq m-1\}$
Therefore $|\,V(G)| = 3p + 6\ and\ |E(G)\,| = 9p$
Define $f: V(G) \to \{2,4, \dots \dots, 2|V(G)|\}$
$f(t_{1,j}) = 2j$
$f(t_{2,j}) = 2(\,j + p + 2)$
$f(t_{3,j}) = 2(j + 2p + 4)$ where $1 \leq j \leq p-1$
$f(s_{1,1}) = 2p$, $f(s_{1,2}) = 2(2p+2)$, $f(s_{1,3}) = 2(3p+4)$

$f\left(s_{2,1}\right)=2p+4, f\left(s_{2,2}\right)=2(2p+4), f\left(s_{2.3}\right)=2(3p+6)+2,$
$f\left(s_{3,1}\right)=2p+2, f\left(s_{3,2}\right)=2(2p+3), f\left(s_{3,3}\right)=2(3p+5)$
Then the edge labeling as follows
$f^*\left(t_{j,k}s_{j,1}\right)=\{1\ if\ \ k\ is\ odd\ 1\leq j\leq 3,1\leq k\leq p-1\ \ 0\ if\ \ k\ is\ even\ 1\leq j\leq 3,1\leq k\leq p-1$
$f^*\left(t_{j,k}s_{j,2}\right)=\{1\ if\ \ k\ is\ odd\ 1\leq j\leq 3,1\leq k\leq p-1\ \ 0\ if\ \ k\ is\ even\ 1\leq j\leq 3,1\leq k\leq p-1$
$f^*\left(t_{j,k}s_{j,3}\right)=\{0\ if\ \ k\ is\ odd\ 1\leq j\leq 3,1\leq k\leq p-1\ \ 1\ if\ \ k\ is\ even\ 1\leq j\leq 3,1\leq k\leq p-1$
$f^*\left(s_{j,1}\,s_{j,2}\right)=1,\ f^*\left(s_{j,2}\,s_{j,3}\right)=0,\ f^*\left(t_{1,1}t_{3,1}\right)=1,\ f^*\left(t_{j,1}t_{j+1,1}\right)=0$, where $1\leq j\leq 2$
Therefore $|e_{f^*}(0)-e_{f^*}(1)|=\left|\frac{9p+1}{2}-\frac{9p-1}{2}\right|\leq 1$
Hence $C\left(3,\underline{\Gamma\left(z_{p^2}\right)}+\Gamma(z_6)\right)$ is ASESDCG.

Example: 3.10 A ASESDCG for $G=C\left(3,\underline{\Gamma(z_{25})}+\Gamma(z_6)\right)$ is given in figure 3.10 where $m=3, p=5$, the vertex set $V(G)=\left\{t_{1,1},t_{1,2},\dots,t_{1,4},t_{2,1},t_{2,2},\dots,t_{2,4},t_{3,1},t_{3,2},\dots,t_{3,4},s_{1,1},s_{1,2},s_{1,3},s_{2,1},s_{2,2},s_{2,3},s_{3,1},s_{3,2},s_{3,3}\right\}$and the edge set $E(G)=\left\{t_{j,k}s_{j,1},t_{j,k}s_{j,2},t_{j,k}s_{j,3},s_{j,1}s_{j,2},s_{j,2}s_{j,3}\ 1\leq j\leq 3,1\leq k\leq 4\right\}\cup\left\{u_{j,1}u_{j+1,1},u_{1,1}u_{3,1}1\leq j\leq 2\right\}$. Therefore $|V(G)|=21$ and $|E(G)\,|=45$. $|e_{f^*}(0)-e_{f^*}(1)|=|23-22|\leq 1$
Hence the graph $C\left(3,\underline{\Gamma(z_{25})}+\Gamma(z_6)\right)$ is ASESDCG.

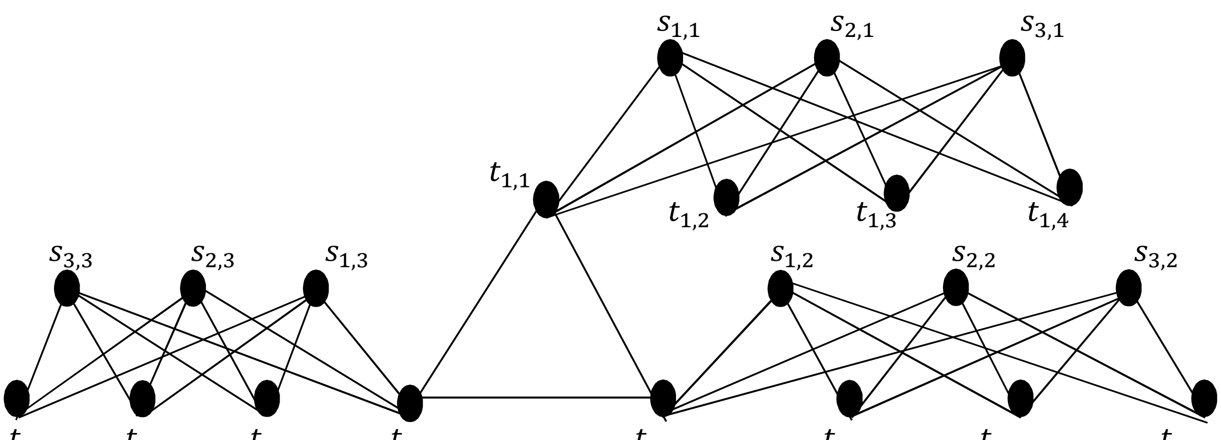

Fig. 3.10 $C\left(3,\underline{\Gamma(z_{25})}+\Gamma(z_6)\right)$

Example: 3.11 A ASESDCG for $G=\ C\left(4,\left(\underline{\Gamma(z_9)}+\Gamma(z_6)\right)\right)$ is given in figure 6 where $m=4, p=3$, the vertex set $V(G)=\ \left\{t_{1,1},t_{1,2},t_{2,1},t_{2,2},t_{3,1},t_{3,2},t_{4,1},t_{4,2}\right\}\cup\left\{s_{1,1},s_{1,2},s_{1,3},s_{1,4},s_{2,1},s_{2,2},s_{2,3},s_{2,4},s_{3,1},s_{3,2},s_{3,3},s_{3,4}\right\}$, the edge set $E(G)=\left\{t_{j,k}s_{j,1},t_{j,k}s_{j,2},t_{j,k}s_{j,3},s_{j,1}s_{j,2},s_{j,2}s_{j,3}\ 1\leq j\leq 4,1\leq k\leq 2\right\}\cup\left\{t_{j,1}t_{j+1,1},t_{1,1}t_{4,1}\right\}1\leq j\leq 3$.
Therefore $|V(G)|=20\ and\ |E(G)\,|=36.\,|e_{f^*}(0)-e_{f^*}(1)|=|20-16|=4\nleq 1$

Hence the graph $G = C\left(4, \underline{\Gamma(z_9)} + \Gamma(z_6)\right)$ does not admit ASESDCG.

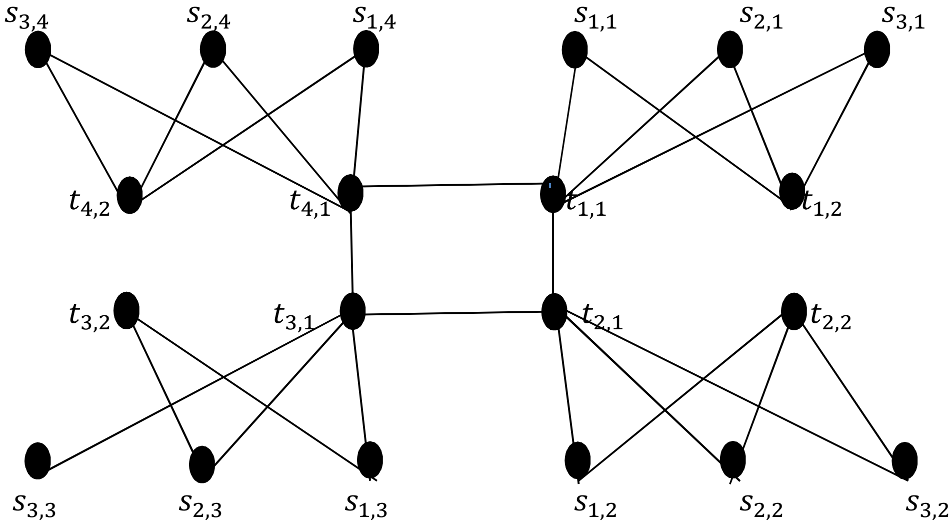

Fig. 3.11 $C\left(4, \underline{\Gamma(z_9)} + \Gamma(z_6)\right)$

Theorem: 3.12 $C\left(m, \underline{\Gamma(z_{p^2})} + \Gamma(z_9)\right)$ where $p \geq 3, m \geq 3$ is ASESDCG.

Proof: Let $G = C\left(m, \underline{\Gamma(z_{p^2})} + \Gamma(z_9)\right)$ be a graph. The vertex set $V(G) =$
$\{t_{1,1}, t_{1,2}, \dots, t_{1,p-1}, t_{2,1}, t_{2,2}, \dots, t_{2,p-1}, \dots t_{m,1}, t_{m,2}, \dots, t_{m,p-1}\} \cup \{s_{1,1}, s_{1,2}, s_{2,1}, s_{2,2}, \dots, s_{m,1}, s_{m,2}\}$
and edge set $E(G) = \{t_{j,k}s_{j,1}, t_{j,k}s_{j,2}, s_{j,1}s_{j,2}, t_{1,1}t_{m,1} 1 \leq j \leq m, 1 \leq k \leq p-1\} \cup \{t_{j,1}t_{j+1,1} 1 \leq j \leq m-1\}$. Therefore $|V(G)| = m(p+1)\ and\ |E(G)| = 2mp$
Define $f: V \to \{2,4, \dots \dots, 2|V(G)|\}$
$f(t_{1,k}) = 2k$
$f(t_{2,k}) = 2(k+p+1)$
$f(t_{3,k}) = 2(k+2p+2)$
.
.
.
$f(t_{m,k}) = 2(k+(m-1)p+(m-1))$ where $1 \leq k \leq p-1$
$f(s_{1,1}) = 2p$
$f(s_{1,2}) = 2(2p+1)$
$f(s_{1,3}) = 2(3p+2)$
.
.
$f(s_{1,m}) = 2(mp+(m-1))$
$f(s_{2,1}) = (p+1)2$
$f(s_{2,2}) = 2(2p+2)$
$f(s_{2,3}) = 2(3p+3)$
.

.

.

$f(s_{2,m}) = 2(mp+m)$

Then the edge labeling as follows

$f^*(t_{j,k}s_{j,1}) = \{1 \ if \ \ k \ is \ odd \ 1 \le j \le m, 1 \le k \le p-1 \ \ 0 \ if \ \ k \ is \ even \ 1 \le j \le m, 1 \le k \le p-1$

$f^*(t_{j,k}s_{j,2}) = \{0 \ if \ \ k \ is \ odd \ 1 \le j \le m, 1 \le k \le p-1 \ \ 1 \ if \ \ k \ is \ even \ 1 \le j \le m, 1 \le k \le p-1$

$f^*(s_{j,1}\, s_{j,2}) = 0, f^*(t_{1,1}t_{m,1}) = 1, \ f^*(t_{j,1}t_{j+1,1}) = 1, where \ 1 \le j \le p-1$

Therefore $|e_{f^*}(0) - e_{f^*}(1)| = |mp - mp| \le 1$

Hence $C\left(m, \underline{\Gamma(z_{p^2})} + \Gamma(z_9)\right)$ is ASESDCG.

Example: 3.13 A ASESDCG for $G = \ C\left(5, \underline{\Gamma(z_9)} + \Gamma(z_9)\right)$ is given in figure 3.13 where $m = 5, p = 3$. The vertex set $V(G) = \{t_{1,1}, t_{1,2}, t_{2,1}, t_{2,2}, t_{3,1}, t_{3,2}, t_{4,1}, t_{4,2}, t_{5,1}, t_{5,2}, s_{1,1}, s_{1,2}, s_{1,3}, s_{1,4}, s_{1,5}, s_{2,1}, s_{2,2}, s_{2,3}, s_{2,4,}s_{2,5}\}$ and edge set $E(G) = \{t_{j,k}s_{j,1}, t_{j,k}s_{j,2}, s_{j,1}s_{j,2}, t_{1,1}t_{5,1} 1 \le j \le 5, 1 \le k \le 2\} \cup \{t_{j,1}u_{j+1,1}\}$**.** Therefore $|V(G)| = 20$ and $|E(G)| = 30$. $|e_{f^*}(0) - e_{f^*}(1)| = |15 - 15| \le 1$

Hence the graph $G = \ C\left(5, \underline{\Gamma(z_9)} + \Gamma(z_9)\right)$ is ASESDCG

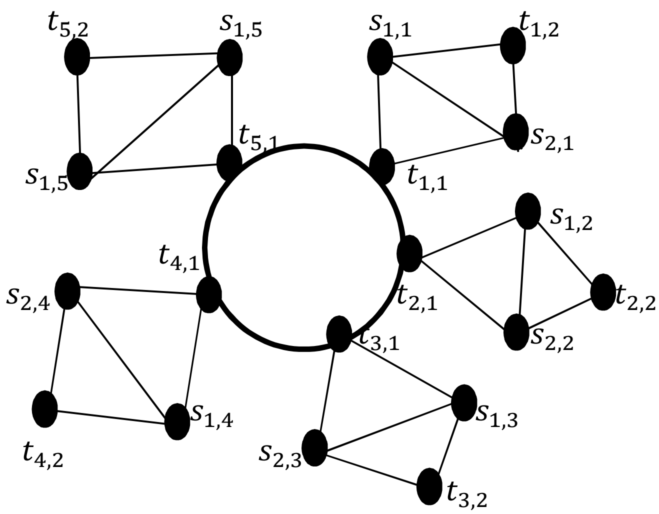

Fig. 3.13 $C\left(5, \underline{\Gamma(z_9)} + \Gamma(z_9)\right)$

Observation:

$i) C\left(m, \Gamma(z_{2p}) + \Gamma(z_4)\right)$ is a ASESDCG ($p \ge 3, m \ge 3$)

$ii)$ $C\left(m, \Gamma(z_{2p}) + \Gamma(z_6)\right)$ is a ASESDCG graph ($p \ge 3, m \ge 3$)

$iii)$ $C\left(m, \Gamma(z_{2p}) + \Gamma(z_9)\right)$ does not admit ASESDCG ($p \ge 3, m \ge 3$)

$iv)$ $C\left(m, \underline{\Gamma(z_{p^2})} + \Gamma(z_4)\right)$ is a ASESDCG ($p \ge 3, m = 3$)

$v) C\left(m, \underline{\Gamma(z_{p^2})} + \Gamma(z_6)\right)$ is a ASESDCG $(p \geq 3, m = 3)$

$vi)\ C\left(m, \underline{\Gamma(z_{p^2})} + \Gamma(z_9)\right)$ is a ASESDCG $(p \geq 3, m \geq 3)$

CONCLUSION:

The study of ASESDCL within zero divisor graphs reveals interesting structural properties and contributes to the broader understanding of graph labeling. We derive several theorems for cycle of join graph of zero divisor graphs. Future research may explore further generalizations and the implications of these labeling in complex network analysis.

REFERENCES:

[1] S. Abhirami, R.Vikrama Prasad, R.Dhavaseelan,"Even Sum Cordial Labeling for Some New Graphs", International Journal of Mechanical Engineering and Technology (IJMET), Volume 9, Issue 2, February 2018, pp. 214-220 Article ID:IJMET_09_02_021, ISSN Print:0976-6340 and ISSN Online:0976-6359

[2] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, J. Algebra, 217, 1999 434–447.

[3] I. Beck, Coloring of commutative rings, J. Algebra, 116(1), 1988 208–226.

[4] Dr. D. Bharathi, M. Lakshmi, "SDC Labeling For Cycle Of Join Zero Divisor Graphs", IOSR Journal of Mathematics (IOSR-JM) e-ISSN: 2278-0661, p-ISSN: 2278-8727, Volume 21, Issue 2, Ser. 1 (Mar. – Apr. 2025), PP 13-20

[5] R.Dhavaseelan, R.Vikrama Prasad and S. Abhirami, "A New Notions of Cordial Labeling Graphs", Global Journal of Pure and Applied Mathematics, ISSN 0973-1768 Volume 11, Number 4 (2015), pp. 1767-1774.

[6] J. A. Gallian, "A Dynamic Survey of Graph Labeling" 1996-2005, The Electronic Journal of Combinatorics.

[7] F. Harary, Graph Theory, Addison-wesley, Reading, Mass 1972.

[8] A. Lourdusamy, S. Jenifer Wency and F. Joy Beaula "SDC Labeling on Zero Divisor Graphs" Sciencia Acta Xaveriana An International Science Journal Vol. 12 No. 2 pp. 11-19 September 2021.

[9] V. Sharma, A. Parthiban, "Average Even Divisor Cordial Labeling: A New Variant Of Divisior Cordial Labeling", TWMS J. App. And Eng. Math. V.14, N.3, 2024, Pp. 1038-1047.

[10] S. K. Vaidya and N. H. Shah, "Further Results on Divisor Cordial Labeling", Annals of Pure and Applied Mathematics Vol. 4, No.2, 2013, 150-159 ISSN: 2279-087X (P), 2279-0888(online).

[11] R. Varatharajan, S. Navanaeethakrishnan, K. Nagarajan, "Divisor Cordial Graphs", International J. Math. Combin. Vol.4(2011), 15-25

[12] Vishally Sharma, A. Parthiban, "On Recent Advances in Divisor Cordial Labeling of Graphs", Mathematics and Statistics 10(1): 140-144, 2022 DOI: 10.13189/ms.2022.100111.