Bounds on Doubly Connected Fair Dominations in Graphs

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Abstract.
A dominating set $\mathbf{D}$ of graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ is a doubly connected fair dominating set (dcfd-set) if it satisfies the following conditions: (1) the induced subgraphs $< \mathbf{D} >$ and $< \mathbf{V}\backslash\mathbf{D} >$ are connected and (2) every vertex not in $\mathbf{D}$ are dominated by the equal number of vertices from $\mathbf{D}$. The dcfd(G) is the minimum cardinality of a dcfd-set in $\mathbf{G}$. In this paper, the doubly connected fair domination number, dcfd(G), of some standard graphs is given. Also, doubly connected fair domination number is related to other parameters. Moreover, Nordhaus-Gaddum-type results are found to be doubly connected fair domination number.

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1.Introduction:

Originally, Ore [14] used the term "domination." Assume that $\mathbf{G}(\mathbf{V}, \mathbf{E})$ is a graph or its domination set. $\mathbf{G}$ is a collection of vertices $\mathbf{D}$ such that any vertex that is not in $\mathbf{D}$ is next to a vertex that is in $\mathbf{D}$. The domination number is represented by $\boldsymbol{\gamma}(\mathbf{G})$. Numerous applications of domination theory exist in both demonology and science [1, 12]. The connected domination number by Sampathkumar and "Walikar"[15]. "Cyman.J" and "Lemanska.M"[6] proposed the doubly connected domination number of graphs. Two-way communication between computer processing and grid vertices was made possible by the notation of doubly connected connection, which sparked a lot of interest in networking [7]. Fair domination in a graph is defined by "Caro.Y", "Hansbery.A" [4, 10], and the fair domination number and minimum cardinality of a fair domination sets are indicated by $\mathbf{fd}(\mathbf{G})$. Moreover, networking used fair domination. "Das.A", "Desormeaux W.J." [8] are used in networks and telecommunication networks, and in addition to being related to other parameters they can frequently provide a connected fair domination on certain standard graphs. To start "Harary F", "Haynes T.W." provide a quick summary of Nordhaus-Gaddum inequalities for the double domination number and several parameters associated with domination [11]. An $\mathbf{p}$-gonal $\mathbf{n}$-cone graph, also called the p-point suspension of $\mathbf{C_p}$ or generalized wheel graph (Buckley and Harary 1988), is defined by the graph join $\mathbf{C_p} + \overline{\mathbf{K}}_\mathbf{n}$, where $\mathbf{C_p}$ is a cyclic graph and $\overline{\mathbf{K}}_\mathbf{n}$ is an empty graph [3]. The case of $k = 3$ pertains to a complete k-partite graph. Specifically, this refers to a tripartite graph, which consists of a collection of vertices divided into three distinct sets, ensuring that no two vertices within the same set are connected. In this structure, every vertex from each set is connected to all vertices in the other two sets. If the three sets contain $\mathbf{m}, \mathbf{p}$, and $\mathbf{q}$ vertices respectively, the complete tripartite graph is represented as $\mathbf{K}_(\mathbf{m}, \mathbf{p}, \mathbf{q})$ [5]. Let $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ be a non-trivial and also simple, connected, undirected graph. If the sub-graphs $< \mathbf{D} >$ produced by $\mathbf{D}$ are connected, then a dominating set $\mathbf{D} \subseteq (\mathbf{G})$ is referred to as a connected dominating set of $\mathbf{G}$. The connected dominating set of $\mathbf{G}$ is represented by $\boldsymbol{\gamma}_\mathbf{c}(\mathbf{G})$. Additionally, the connected dominating set satisfies that all vertex in $\mathbf{V}\backslash\mathbf{D}$ has an equal number of adjacent in $\mathbf{D}$. The notation $\mathbf{cfd}(\mathbf{G})$ represents the connected fair domination number of $\mathbf{G}$. If a set $\mathbf{D} \subseteq (\mathbf{G})$ is both dominating and connected at both $< \mathbf{D} >$ and $< \mathbf{V}\backslash\mathbf{D} >$, then it is doubly connected. The doubly connected domination number is represented by the symbol $\boldsymbol{\gamma}_\mathbf{cc}(\mathbf{G})$. If every vertex in $\mathbf{V}\backslash\mathbf{D}$ is dominated by an equal number of vertices from $\mathbf{D}$, then set $\mathbf{D} \subseteq \mathbf{V}(\mathbf{G})$ is a fair dominating set. The symbol $\mathbf{fd}(\mathbf{G})$ represents the fair domination number of $\mathbf{G}$. If (1) $\mathbf{D}$ dominates $\mathbf{G}$, (2) $< \mathbf{D} >$ and $< \mathbf{V}\backslash\mathbf{D} >$ are connected, and (3) for each $v \in \mathbf{D}$, $|N(v) \cap \mathbf{D}| = k$, then a set $\mathbf{D} \subseteq \mathbf{V}$ is called a doubly connected k-fair domination set, or simply kdcfd-set, for a given k-dcfd-set in $\mathbf{G}$, the lowest cardinality is its doubly connected k-fair domination, number denoted as $dcfd_k(\mathbf{G})$. When an integer $k \geq 1$, output: a doubly connected fair dominating set in $\mathbf{G}$ is represented as a kdcfd-set. A graph $\mathbf{G}$, cardinality is represented by the dcfd number or dcfd(G). In other words, if k is an integer, then $dcfd(\mathbf{G}) = \min dcfd_k$, where the minimum is found over all integers $1 \leq k \leq n-1$. Thus, the following conclusions are drawn.

2.Definition:

A dominating set $\mathbf{D}$ of a graph $\mathbf{G}(\mathbf{V}, \mathbf{E})$ is a dcfd-set if it satisfies (1) all vertices not in $\mathbf{D}$ are dominated by the same number of vertices from $\mathbf{D}$ (2) every induced subgraph $< \mathbf{D} >$ and $< \mathbf{V} \backslash \mathbf{D} >$ are connected. The minimum cardinality of the dcfd-set in $\mathbf{G}$ is the dcfd number is denoted by $dcfd(\mathbf{G})$.

Proposition 1. *Support vertices are elements of each dcfd-set.*

Proof:

Let $\mathbf{v}$ and $\mathbf{w}$ be any vertices in $\mathbf{G}$ such that $deg(\mathbf{v}) = 1$ and $deg(\mathbf{w}) \geq 1$. Thus $\mathbf{v}$ is adjacent to $\mathbf{w}$ because $\mathbf{w}$ is a support vertex of $\mathbf{v}$. If $\mathbf{w} \in \mathbf{V} \backslash \mathbf{D}$. Then $\mathbf{w}$ is adjacent to some vertex in $\mathbf{D}$ and $\mathbf{v} \in \mathbf{D}$. Therefore $\mathbf{V} \backslash \mathbf{D}$ is not connected, which is a contradiction because $< \mathbf{D} >$ and $< \mathbf{V} \backslash \mathbf{D} >$ are connected. Thus $\mathbf{w} \in \mathbf{D}$.

Proposition 2. *Let* $\mathbf{G} \cong \mathbf{W_p} \cup \mathbf{u}$ *be a wheel with* $\mathbf{p}$ *vertices, and* $\mathbf{u}$ *is a pendant vertex, then* $dcfd(\mathbf{G}) = \begin{cases} 2. If\ \mathbf{u}\ is\ a\ pendant\ of\ a\ center\ vertex\ in\ \mathbf{G} \\ \mathbf{P-1}, otherwise \end{cases}$

Proof:

Let $\mathbf{G}(\mathbf{V}, \mathbf{E})$ be a wheel with $\mathbf{p}$ vertices, namely $\mathbf{p_1}, \mathbf{p_2}, \ldots \mathbf{p_i}, \mathbf{p_j}, \ldots \mathbf{p_n}$ with all $deg(\mathbf{p_i}) = 3$ and $deg(\mathbf{p_j}) = \mathbf{p-1}$ where $\mathbf{p_j}$ is the center vertex in $\mathbf{G}$, and $\mathbf{i} \neq \mathbf{j}$, $\mathbf{1} \leq \mathbf{i}, \mathbf{j} \leq \mathbf{p}$

case 1: If $\mathbf{u}$ is a pendant of a center vertex $\mathbf{p_j}$ in $\mathbf{G}$, then dcfd-set $\mathbf{D} = \{\mathbf{u}, \mathbf{p_j}\}$. Therefore, $dcfd(\mathbf{G}) = 2$.

case 2: If $\mathbf{u}$ is a pendant of any other vertices $\mathbf{p_i} \in \mathbf{G}$ and the vertex set becomes $\mathbf{V_1} = \mathbf{V} \cup \{\mathbf{u}\}$ then dcfd-set $\mathbf{D} = (\mathbf{V_1} - \{\mathbf{p_R}, \mathbf{p_r}\})$ where $1 \leq \mathbf{R}, \mathbf{r} \leq \mathbf{p}$ and $\mathbf{R} \neq \mathbf{r}$. Therefore

$$dcfd(\mathbf{G}) = \mathbf{p} - 2 + 1 = \mathbf{p} - 1.$$

Proposition 3. *If* $\mathbf{G} \cong \mathbf{W_p} \cup \mathbf{u}, \mathbf{w}$ *be a wheel on* $\mathbf{p}$ *vertices with a pendant* $\{\mathbf{u}, \mathbf{w}\}$, *then* $dcfd(\mathbf{G}) = \mathbf{p}$.

Proof:

Given $\mathbf{G}$ is a wheel on $\mathbf{p}$ vertices, namely $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}$, with all $deg(\mathbf{p_i}) = 3$ and $deg(\mathbf{p_j}) = \mathbf{p-1}$ where $\mathbf{p_j}$ is the center vertex in $\mathbf{G}$ and $\mathbf{i} \neq \mathbf{j}$, $1 \leq \mathbf{i}, \mathbf{j} \leq \mathbf{p}$. If $\mathbf{u}, \mathbf{w}$ is a pendant of a non-adjacent vertex in $\mathbf{G}$ and also $\mathbf{u}$ is a pendant of center vertices $\mathbf{p_j}$ and $\mathbf{w}$ is pendant of any other vertices in $\mathbf{p}$, then dcfd-set $\mathbf{D} = (\mathbf{V} - \{\mathbf{p_m}, \mathbf{p_l}\})$ where $\mathbf{p_m}$ and $\mathbf{p_l}$ are adjacent, $|\mathbf{N}(\mathbf{p_m}, \mathbf{p_l}) \cap \mathbf{D}| = \mathbf{k}$, and $1 \leq \mathbf{m}, l \leq \mathbf{n}$, $\mathbf{m} \neq l$. Therefore, $dcfd(\mathbf{G}) = \mathbf{p} - 2 + 2$. $dcfd(\mathbf{G}) = \mathbf{p}$.

Proposition 4. *Let* $\mathbf{G}$ *be a non-trivial and also connected graph with two pendant vertices. Then* $dcfd(\mathbf{G}) \leq \mathbf{p} + 1$.

Proof:

Given $\mathbf{G}$ is a connected and also non-trivial graph with two pendant vertices. Consider $\mathbf{G} \cong \mathbf{T_p} \cup \mathbf{u}, \mathbf{w}$, where $\mathbf{T}$ is a tree on $\mathbf{p}$ vertices denoted as $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_p}$, and $\{\mathbf{u}, \mathbf{w}\}$ is a pendant of any other vertex in $\mathbf{T}$. We know that dcfd-set of $\mathbf{T}$ is $\mathbf{p} - 1$. Thus dcfd-set $\mathbf{D} = (\mathbf{V} - \{\mathbf{p_k}\})$ where $< \mathbf{D} >$ and $< \mathbf{V} \backslash \mathbf{D} >$ are adjacent, $|\mathbf{N}(\mathbf{p_k}) \cap \mathbf{D}| = \mathbf{k}$. Therefore, $dcfd(\mathbf{G}) = \mathbf{p} - 1 + 2$

$dcfd(\mathbf{G}) \leq \mathbf{p} + 1$.

Proposition 5. *Let* $\mathbf{G} \cong \mathbf{K_{m,p}}$ *represent a complete bipartite made up of* $m + p$ *vertices* $(\mathbf{m}, \mathbf{p} \geq \mathbf{2})$ *with a pendant* $\{\mathbf{u}, \mathbf{w}\}$ *then*

$$dcfd(\mathbf{G}) = \begin{cases} 4, If\ \mathbf{u}, \mathbf{w}\ is\ pendant\ of\ any\ adjacent\ vertex\ in\ \mathbf{G} \\ 6, otherwise \end{cases}$$

Proof:

Consider $\mathbf{G}$ as a fully connected bipartite graph with $\mathbf{m} + \mathbf{p}$ vertices, namely $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_m}, \mathbf{q_1}, \mathbf{q_2}, \ldots, \mathbf{q_p}$, where $\mathbf{m} \neq \mathbf{p}$ and $\mathbf{m}, \mathbf{p} \geq 2$ with two pendant vertices $\{\mathbf{u}, \mathbf{w}\}$. Degree of $\mathbf{p_i}$ is $\mathbf{p}$ and degree of $\mathbf{q_j}$ is $\mathbf{m}$

case 1: If $\mathbf{u}$ and $\mathbf{w}$ are pendant vertices of any other adjacent vertices in $\mathbf{G}$, the vertex set $\mathbf{V} = \mathbf{p_1}, \ldots, \mathbf{p_m}, \mathbf{q} - \mathbf{1}, \ldots, \mathbf{q_p}, \mathbf{u}, \mathbf{w}$. We know that $dcfd(\mathbf{K_{m,p}}) = 2$ where $\mathbf{m} \neq \mathbf{p}$,then dcfd-set contains support and pendant vertices $\mathbf{D} = \{\mathbf{p_i}, \mathbf{q_j}, \mathbf{u}, \mathbf{w}\}$ are adjacent, and each vertex in $\mathbf{D}$ is adjacent to an identical number of vertices in $\mathbf{V} \backslash \mathbf{D}$.Thus $dcfd(\mathbf{G}) = 4$

case 2: If $\mathbf{u}, \mathbf{w} \in \mathbf{G}$, where $\mathbf{u}$ and $\mathbf{w}$ are represented as pendant vertices of any non-adjacent vertices in $\mathbf{G}$. Let $\mathbf{G} \in \mathbf{K_{m,p}} \cup \mathbf{u}, \mathbf{w}$, where $\mathbf{K_{m,p}}$ is denoted as complete bipartite with two partitions into the vertex set. $\mathbf{m}, \mathbf{p}$ $(\mathbf{m} \neq \mathbf{p})$ when each vertex in $\mathbf{m}$ is connected to each vertex in $\mathbf{p}$. Let $\mathbf{u}$ and $\mathbf{w}$ be pendant of any non-adjacent vertices in $\mathbf{m}$ or $\mathbf{p}$. Then dcfd-set $\mathbf{D} = \{\mathbf{u}, \mathbf{w}, \mathbf{p_i}, \mathbf{p_l}, \mathbf{q_j}, \mathbf{q_k}\}$ is connected, and $< \mathbf{V} \backslash \mathbf{D} >$ also connected,$|\mathbf{N}(\mathbf{V} \backslash \mathbf{D}) \cap \mathbf{D})| = 2$. Therefore, $dcfd(\mathbf{G}) = 6$

2.1 Observation

Let $\mathbf{G}$ be a connected, nontrivial graph.

1. $\boldsymbol{\gamma_c}(\mathbf{G}) \leq \mathbf{cfd}(\mathbf{G}) \leq dcfd(\mathbf{G})$ For any graph $\mathbf{G}$.
2. $\mathbf{fd}(\mathbf{G}) \leq \mathbf{cfd}(\mathbf{G}) \leq dcfd(\mathbf{G})$ For any graph $\mathbf{G}$.
3. $dcfd(\mathbf{P_p}) = \mathbf{p-1}$, where $\mathbf{P_p}$ is a path on $\mathbf{p}$ vertices for all $\mathbf{p} \geq 2$.
4. $dcfd(\mathbf{C_p}) = \mathbf{p} - 2$, where $\mathbf{C_p}$ is a cycle graph on $\mathbf{p}$ vertices for all $\mathbf{p} \geq 3$.
5. $dcfd(\mathbf{K_p}) = 1$ and $dcfd(\mathbf{K_{m,p}}) = 2$, where $\mathbf{K_p}$ and $\mathbf{K_{m,p}}$ are complete graph and complete bipartite graph respectively where $\mathbf{p} \geq 2$, $\mathbf{m} \geq 2$
6. $dcfd(\mathbf{K_{1,p-1}}) = \mathbf{p-1}$, where $\mathbf{K_{1,p-1}}$ is a star graph with $\mathbf{p}$ vertices for $\mathbf{p} \geq 2$.

Theorem 2.1. *For a complete bipartite graph* $\mathbf{K_{m,p}}$ *of* $\mathbf{m+p}$ *vertices, where* $\mathbf{m+p} \geq 1$ $dcfd(\mathbf{K_{m,p}}) = \begin{cases} \mathbf{p}, for\ \mathbf{m} = 1\ and\ \mathbf{p} \geq 2 \\ 2, for\ \mathbf{m}, \mathbf{p} \geq 2 \end{cases}$

Proof:

Given $\mathbf{G}$ be a complete bipartite graph with disjoint sets $(\mathbf{U_m}, \mathbf{V_p})$ that has $\mathbf{m+n}$ vertices, namely $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}, \mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}$. with all $deg(\mathbf{u_i}) = \mathbf{p}$ and $deg(\mathbf{v_j}) = \mathbf{m}$, where $\mathbf{i} \neq \mathbf{j}$, $1 \leq \mathbf{i}$, $\mathbf{j} \leq \mathbf{p}$

case 1: Let $\mathbf{m} = \mathbf{1}$ and $\mathbf{p} \geq 2$. Then $\mathbf{G} = \mathbf{K_{1,p}}$ is a star of order $\mathbf{p}+1$. by observation (6) $dcfd(\mathbf{K_1}, \mathbf{p}) = \mathbf{p}$.

case 2: Let $\mathbf{m}, \mathbf{p} \geq 2$, then $\mathbf{G} = \mathbf{K_{m,p}}$ of order $\mathbf{m+p}$. Consider $\mathbf{D} = \mathbf{u_i}$, where $\mathbf{u_i} \in \mathbf{m}$, $deg(\mathbf{u_i}) = \mathbf{p}$; thus. $\mathbf{u_i}$ is not adjacent to $\mathbf{U_m}$. Therefore, dcfd-set $\mathbf{D} = \{\mathbf{u_i}, \mathbf{v_j}\}$ where $deg(\mathbf{u_i}) = \mathbf{m}$ and $deg(\mathbf{v_j}) = \mathbf{p}$ where $\mathbf{i} \neq \mathbf{j}$, $1 \leq \mathbf{i}, \mathbf{j} \leq \mathbf{p}$. $\mathbf{u_i}$ is adjacent to all vertices in $\mathbf{V_p}$, and $< \mathbf{V} \backslash \mathbf{D} >$ are connected. Thus, $dcfd(\mathbf{K_{m,p}}) = 2$ for all $\mathbf{m}, \mathbf{p} \geq 2$.

Theorem 2.2. *For a complete tripartite graph* $\mathbf{K_{m,p,q}}$ *of order* $\mathbf{m+p+q}$ *where* $\mathbf{p}, \mathbf{q}, \mathbf{m} \geq 1$

$$dcfd(\mathbf{K_{m,p,q}}) = \begin{cases} 1, If\ \mathbf{m} = \mathbf{p} = \mathbf{q} = \mathbf{1} \\ 1, where\ \mathbf{m} = 1\ and\ \mathbf{q}, \mathbf{p} \geq 2 \\ 3, otherwise \end{cases}$$

Proof:

Let $\mathbf{G} = \mathbf{K_{m,p,q}}$ be a complete tripartite with partition $(\mathbf{U_m}, \mathbf{V_p}, \mathbf{W_q})$, where $\mathbf{U_m} = \mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}$; $\mathbf{V_p} = \mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}$. and $\mathbf{W_q} = \mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_q}$ with $deg(\mathbf{u_i}) = \mathbf{p+q}$, $deg(\mathbf{v_j}) = \mathbf{m+q}$, and $deg(\mathbf{w_k}) = \mathbf{m+p}$, and $\mathbf{i} \neq \mathbf{j} \neq \mathbf{k}$, $1 \leq \mathbf{i}, \mathbf{j}, \mathbf{k} \leq \mathbf{p}$

Case 1: Suppose that $\mathbf{m} = \mathbf{p} = \mathbf{q} = \mathbf{1}$; then $\mathbf{G} = \mathbf{K_{1,1,1}}$, a cycle graph of order 3, is connected. Then dcfd-set $\mathbf{D} = \{\mathbf{u_i}\}$. Therefore, $dcfd(\mathbf{G}) = 1$.

Case 2: If $\mathbf{q}, \mathbf{p} \geq 2$ and $\mathbf{m} = 1$. Then $\mathbf{G} = \mathbf{K_{1,q,p}}$, If $\mathbf{q}, \mathbf{p} = 2$ and $\mathbf{m} = 1$, then $deg(\mathbf{u_i}) = 4$, $deg(\mathbf{v_j}) = 3$, and $deg(\mathbf{w_k}) = 3$ where $\mathbf{u_i}$ is adjacent to $\mathbf{V_p}, \mathbf{W_q}$, then dcfd-set $\mathbf{D} = \{\mathbf{u_i}\}$. Thus $dcfd(\mathbf{K_{1,q,p}}) = 1$. Let $\mathbf{m} = 1$ and $\mathbf{p}, \mathbf{q} \geq 2$ with all $deg(\mathbf{u_i}) = \mathbf{p+q}$, $deg(\mathbf{v_j}) = \mathbf{1+q}$, and $deg(\mathbf{w_k}) = \mathbf{1+p}$. Since $\mathbf{u_i}$ is adjacent to $\mathbf{V_p}, \mathbf{W_q}$ in $\mathbf{G}$ then dcfd-set $\mathbf{D} = \{\mathbf{u_i}\}$. Therefore, $dcfd(\mathbf{G}) = 1$

Case 3: Let $\mathbf{m}, \mathbf{p}, \mathbf{q} \geq 2$, then $\mathbf{G} = \mathbf{K_{m,p,q}}$, where $|mathbfu_i$ is not adjacent to $\mathbf{U_m}$ because $(\mathbf{U_m}, \mathbf{V_p}, \mathbf{W_q})$ are null set by definition of complete tripartite one null set is adjacent to the other two null sets, and the vertex set becomes $\mathbf{V_1} = \{(\mathbf{U_m} + \mathbf{V_p} + \mathbf{W_q})\}$ where $1 \leq \mathbf{m}, \mathbf{p}, \mathbf{q} \leq \mathbf{p}$, then dcfd-set $\mathbf{D} = \{\mathbf{u_i}, \mathbf{v_j}, \mathbf{w_k}\}$. Thus $dcfd(\mathbf{G}) = 3$ for $\mathbf{m}, \mathbf{p}, \mathbf{q} \geq 2$.

Theorem 2.3. *For a cone graph* $\mathbf{C_p} + \overline{\mathbf{K_n}}$ *of order* $\mathbf{p+n}$

$$dcfd(\mathbf{C_p} + \overline{\mathbf{K_n}}) \leq \begin{cases} 1, whenever\ \mathbf{p} \geq 3\ and\ \mathbf{n} = 1 \\ \mathbf{p+n} - 2, whenever\ \mathbf{p}, \mathbf{n} \geq 3 \end{cases}$$

Proof:

Let $\mathbf{G} = \mathbf{C_p} + \overline{\mathbf{K_n}}$, where $\mathbf{C_p}$ is a cycle on $\mathbf{p}$, which is a $(\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_p})$ vertices, and $\overline{\mathbf{K_n}}$ is a null graph on $\mathbf{n}$, which is a $(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}.)$ vertices with all $deg(\mathbf{u_j}) = \mathbf{2+n}$ and $deg(\mathbf{v_i}) = \mathbf{p}$ where $\mathbf{i} \neq \mathbf{j}$. If $\mathbf{G}$ contains $\mathbf{p+n}$ vertices. Then consider the following case.

Case 1: Suppose that $\mathbf{p} \geq 3$ and $\mathbf{n} = 1$. Then $\mathbf{G} = \mathbf{C_p} + \overline{\mathbf{K_1}}$, a wheel graph of order $\mathbf{p}+1$ with $deg(\mathbf{u_j}) = 3$ and $deg(\mathbf{v_i}) = \mathbf{m}$, then dcfd-set $\mathbf{D} = \{\mathbf{v_i}\}$. Since all vertices in $< \mathbf{V(G)} \backslash \mathbf{D} >$ are adjacent to $\mathbf{v_i}$, in $\mathbf{G}$. Thus $dcfd(\mathbf{C_m} + \overline{\mathbf{K_1}}) = 1$

Case 2: Suppose that $\mathbf{m}, \mathbf{n} \geq 3$. Then $\mathbf{G} = \mathbf{C_m} + \overline{\mathbf{K_n}}$, consider $\mathbf{D} = \{\mathbf{a}\}$, where $\mathbf{a} \in \overline{\mathbf{K_n}}$. Since $\overline{\mathbf{K_n}}$ is empty graph. Thus every vertex of $\mathbf{C_m}$ is adjacent to $\mathbf{D}$, but $\mathbf{D}$ is not adjacent to other vertices in $\overline{\mathbf{K_n}}$. Therefore one cannot be the $dcfd(\mathbf{C_m} + \overline{\mathbf{K_n}})$, where $\mathbf{m}, \mathbf{n} \geq 3$. Such that $\mathbf{m} = 4{,}6$ and $\mathbf{n} \geq 2$, next choose $\mathbf{a}, \mathbf{b} \in \mathbf{C_m}$, and $\mathbf{u} \in \overline{\mathbf{K_n}}$, $\mathbf{D} = \mathbf{a}, \mathbf{b}, \mathbf{u}$ are connected; the element in $\overline{\mathbf{K_n}}$ is adjacent to each vertex in $\mathbf{C_m}$.Since $< \mathbf{V(G)} \backslash \mathbf{D} >$ are connected. If $\mathbf{u} \in \mathbf{V(G)} \backslash \mathbf{D}$, $|\mathbf{N(u)} \cap \mathbf{D}| = 2$. Hence, $dcfd(\mathbf{C_m} + \overline{\mathbf{K_n}}) = 3$, where $\mathbf{m} = 4{,}6$, and $\mathbf{n} \geq 2$. Next, pick 5 elements from $\mathbf{C_m}$ and two from $\overline{\mathbf{K_n}}$, where $\mathbf{m} = 7{,}9$ and $\mathbf{n} \geq 2$. $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{C_m}$ and $\mathbf{u}, \mathbf{v} \in \overline{\mathbf{K_n}}$

3.Bounds on $dcfd(\mathbf{G})$ in Terms of The Order of G

Theorem 3.1. *For any connected graph* $\mathbf{G}$, $dcfd(\mathbf{G}) \leq 2\mathbf{p} - 1$.

Proof:

$\mathbf{G}$ is complete with two vertices, then $dcfd(\mathbf{G}) = 1$. Let $\mathbf{n} \geq 3$, consider $\mathbf{G} = \mathbf{F_p}$ containing $2\mathbf{p}+1$. vertices represented as $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_{2p+1}}$. Choose $\mathbf{p_i}, \mathbf{p_j} \in \mathbf{V} \backslash \mathbf{D}$, where $\mathbf{p_i}$ and $\mathbf{p_j}$ are connected. The dcfd-set $\mathbf{D_1} = \mathbf{V} \backslash \{\mathbf{p_1}, \mathbf{p_2}\}$. As $\mathbf{D_1}$ contains $\mathbf{D}$, $\mathbf{D_1}$ is a dominating set of $\mathbf{G}$. $|\mathbf{N}(\mathbf{p_i}, \mathbf{p_j}) \cap \mathbf{D}| = 1$. Then dcfd-set $\mathbf{D} = \mathbf{V} - \{\mathbf{p_i}, \mathbf{p_j}\}$. Therefore, $dcfd(\mathbf{G}) \leq 2\mathbf{p} + 1 - 2$.

$dcfd(\mathbf{G}) \leq 2\mathbf{p} - 1$.

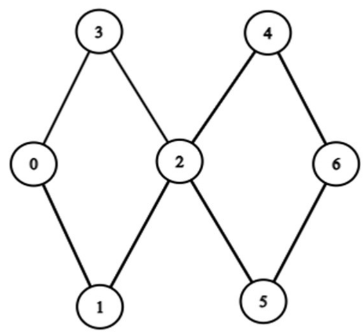
Remark 3.1. The above theorem is sharp when $\mathbf{G} = \mathbf{F_p}$, where $\mathbf{F_p}$ is a friendship graph. For, $\mathbf{F_p}$ is a collection of $\mathbf{p}$ triangles with a common vertex.

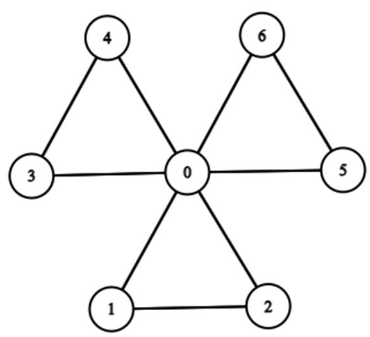
Remark 3.2. The above theorem bounds are stringent. Consider $figure\ (a)$; clearly, $dcfd(\mathbf{G}) = 5$. Check the $dcfd(\mathbf{G})$ in $figure\ (b)$ percent a formal proof. Consider the $fig\ (b)$ the graph $\mathbf{G}$ has 3 vertices. But the graph $\mathbf{G}$ is a friendship graph. So it has $2\mathbf{p}+1$ vertices and $3\mathbf{p}$ edges. Thus, the graph $\mathbf{G}$ has 7 vertices and 9 edges. Prove that $dcfd(\mathbf{G}) = 5$.

case 1: If the dominating set of graph $\mathbf{G}$ is 0, it's maintaining all vertices $\mathbf{V} \backslash \mathbf{D}$ adjacent to vertices in $\mathbf{D}$.

case 2: If $0 \in \mathbf{fd(G)}$, then 1, 2, 3, 4, 5, and 6 are dominant 1 time, respectively. So 0 is also a $\mathbf{cfd}$ set. That $\mathbf{cfd}$-set is size 1. But it is not doubly connected.

case 3: If $0 \in \mathbf{fd}$, then $\mathbf{V \backslash D}$ are not connected. Now add another element to the $\mathbf{cfd}$-set, $0,1 \in \boldsymbol{\gamma}_{\mathbf{c}}(\mathbf{G})$. Then 2, 3, 4, 5, and 6 dominate 2, 1, 1, 1, and 1 times, respectively. So 0, 1 is not a $\mathbf{cfd}$-set. Include another element. Thus 0, 1 and 2 are in the $\boldsymbol{\gamma}_{\mathbf{c}}(\mathbf{G})$-set, and then 3, 4, 5, and 6 are dominated 1 time. Therefore, $\{0,1,2\}$ is a $\mathbf{cfd}$-set but $\mathbf{V \backslash D}$ are not adjacent. Add another element to the $\mathbf{cfd}$ set. 0, 1, 2, and 3, which is a connected set. Then 4, 5, and 6 dominate 2, 1, and 1. Thus $\{0,1,2,3\}$ is not a $\mathbf{cfd}$ set of size 4.
case 4: If $0,1,2,3 \in \boldsymbol{\gamma}_{\mathbf{c}}(\mathbf{G})$-set include another element, then 0, 1, 2, 3, 4, are connected, and then 5, 6, are dominated 1 time, respectively. $\{0,1,2,3,4\} \in \mathbf{cfd}set$. Now $\{5,6\} \in \mathbf{V \backslash D}$ are connected. Therefore $\{0,1,2,3,4\} \in dcfd-set$ and dcfd-set of size 5. All cases get $dcfd(\mathbf{G}) = 5$ and hence the result.





(a)
(b)

Theorem 3.2. *For any connected regular graph* $\mathbf{G}$ *with* $\mathbf{p} \geq 3$ *vertices. Then* $dcfd(\mathbf{G}) \leq \mathbf{p} - 2$.
Proof:
Given $\mathbf{G}$ be a r-regular connected graph, let $\mathbf{D}$ be a minimum doubly connected domination set of $\mathbf{G}$. Then $|\mathbf{D}| \leq \mathbf{p} - 2$. Choose $\mathbf{p_1}, \mathbf{p_2} \in \mathbf{V \backslash D}$, where $\mathbf{p_1}$ and $\mathbf{p_2}$ are adjacent and the set $\mathbf{D_1} = \mathbf{V \backslash p_1, p_2}$. As $\mathbf{D} \subseteq \mathbf{D_1}$, $\mathbf{D_1}$ dominates $\mathbf{G}$. Let $\mathbf{x}, \mathbf{y} \in \mathbf{D_1}$. Since $\mathbf{x}$ and $\mathbf{y}$ are either in $\mathbf{D}$ or adjacent to the vertex in $\mathbf{D}$ and $\mathbf{p_1}, \mathbf{p_2} \in \mathbf{V \backslash D}$: both $\mathbf{p_1}$ and $\mathbf{p_2}$ are connected in $\mathbf{V \backslash D}$, or there is a path between $\mathbf{p_1}$ and $\mathbf{p_2}$ in $\mathbf{V \backslash D}$. Then $< \mathbf{D_1} >$ is doubly connected and also satisfies $\mathbf{p_1}, \mathbf{p_2} \in \mathbf{V \backslash D}$; the neighbors of $\mathbf{p_1}$ and $\mathbf{p_2}$ are connected to an equal number of vertices in $\mathbf{D}$, $|\mathbf{N}(\mathbf{p_1}, \mathbf{p_2}) \cap \mathbf{D}| = \mathbf{k}$. Thus, $< \mathbf{D} >$ is a doubly connected fair domination set of $\mathbf{G}$, thus $dcfd(\mathbf{G}) \leq \mathbf{p} - 2$
Remark 3.3. The above theorem is sharp when $\mathbf{G} \cong \mathbf{C_p}$, where $\mathbf{C_p}$ is a cycle graph with $\mathbf{p} \geq 3$ vertices.
Theorem 3.3. $\mathbf{G}$ *be a 3-connected graph of* $\mathbf{p}$ *vertex. Then* $dcfd(\mathbf{G}) \leq \mathbf{p} - 3$.
Proof:
Assume that $\mathbf{G}$ is denoted as a 3-connected graph of order $\mathbf{p}$. Clearly, $\delta > 3$. Now, in the graph $\mathbf{G}$, there are at least three vertices such that $\mathbf{deg}(\mathbf{p}) = \mathbf{deg}(\mathbf{q}) = \mathbf{deg}(\mathbf{r}) = \mathbf{k}$, where $3 \leq \mathbf{k} \leq \mathbf{p} - 1$. Let $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathbf{V \backslash D}$, where $\mathbf{p}, \mathbf{q}$, and $\mathbf{r}$ are adjacent vertices or there is a path between $\mathbf{p}, \mathbf{q}$, and $\mathbf{r}$ in $\mathbf{V \backslash D}$. Let $\mathbf{D} = \mathbf{V \backslash p, q, r}$. Since $\mathbf{G}$ is 3-connected and all vertex in $\mathbf{D}$ is connected, therefore $< \mathbf{D} >$ is the dcd set of $\mathbf{G}$. Also, $\mathbf{p}, \mathbf{q}$, and $\mathbf{r}$ are adjacent to $\mathbf{k}$ or fewer than $\mathbf{k} - 1$ vertices in $\mathbf{D}$. Therefore, $< \mathbf{D} >$ is a dcfd set in $\mathbf{G}$. Thus, $dcfd(\mathbf{G}) \leq \mathbf{p} - 3$.
Remark 3.4. The above theorem is sharp when $\mathbf{G} \cong \mathbf{K_p}$, where $\mathbf{K_p}$ is a complete graph with $\mathbf{p} = 4$.
Theorem 3.4. *Let* $\mathbf{T}$ *be a tree with* $\mathbf{p} \geq 3$ *vertices. Then* $dcfd(\mathbf{T}) = \mathbf{p} - 1$
Proof:
Let $\mathbf{D}$ be a $dcfd(\mathbf{T})$-set. Then $\mathbf{p_1} \in \mathbf{V \backslash D}$ contains pendant vertex, say $\mathbf{p_1}$, and $deg(\mathbf{p_1}) = 1$. Therefore $\mathbf{D} = \mathbf{V \backslash u}$. Since both sub-graph $< \mathbf{D} >$ and $< \mathbf{V \backslash D} >$ are connected and also $\mathbf{u}$ is adjacent, to the vertex in $\mathbf{D}$ ($|\mathbf{N}(\mathbf{u}) \cap \mathbf{D}| = \mathbf{1}$). Thus, $\mathbf{D}$ is a dcfd-set of $\mathbf{G}$, and hence $dcfd(\mathbf{G}) = \mathbf{p} - 1$.
Corollary 1. *Assume* $\mathbf{T}$ *be a tree with* $\mathbf{p} \geq 3$. *Then* $\boldsymbol{\gamma}_{\mathbf{cc}}(\mathbf{T}) = dcfd(\mathbf{T})$.
Theorem 3.5. *For any graph* $\mathbf{G}$ *which is connected with* $\mathbf{q}$ *edges and* $\mathbf{p}$ *vertices.*
Then $\frac{\mathbf{p}}{\Delta(\mathbf{G})+1} \leq dcfd(\mathbf{G}) \leq \mathbf{q}$
Proof:
The lower bound results from the fact that $\boldsymbol{\gamma}_{\mathbf{cc}}(\mathbf{G}) \geq \frac{\mathbf{p}}{\Delta(\mathbf{G})+1}$ and $\boldsymbol{\gamma}_{\mathbf{cc}}(\mathbf{G}) < dcfd(\mathbf{G})$ First, observe that for a connected graph when discussing the upper bound $\mathbf{q} \geq \mathbf{p} - 1$. $\mathbf{G}$ is a tree if $\mathbf{q} = \mathbf{p} - 1$ in a connected graph. And in such case $dcfd(\mathbf{G}) = \mathbf{p} - \mathbf{1} = \mathbf{q} + 1 - 1$. Now $dcfd(\mathbf{G}) < \mathbf{q}$. Hence $\frac{\mathbf{p}}{\Delta(\mathbf{G})+1} \leq dcfd(\mathbf{G}) \leq \mathbf{q}$

4.Nordhus-Gaddum Type Results on $dcfd$-Set of G

Theorem 4.1. *For any connected graph* $\mathbf{G}$ *and* $\overline{\mathbf{G}}$*, then* $6 \leq dcfd(\mathbf{G}) + dcfd(\overline{\mathbf{G}}) \leq 2\mathbf{p} - 2$

Proof:

Given $\mathbf{G}$ and $\overline{\mathbf{G}}$ are connected. If $\mathbf{p} = 2$, then the complement graph $\overline{\mathbf{G}}$ is disconnected. According to the result, the final outcome, is obtained with $\mathbf{p} \geq 4$ vertices in $\mathbf{G}\&\overline{\mathbf{G}}$, both of which are connected. The lower bound of the sum, if $\mathbf{p} \geq 4$, then $dcfd(\mathbf{G}) = 3$ and $dcfd(\overline{\mathbf{G}}) = 3$; hence $dcfd(\mathbf{G}) + dcfd(\overline{\mathbf{G}}) = 6$; therefore, $dcfd(\mathbf{G}) + dcfd(\overline{\mathbf{G}}) \geq 6$. Let $\mathbf{D}$ be a dcfd-set of $\mathbf{G}$; then $dcfd(\mathbf{G}) \leq 2\mathbf{p} - 1$ for any connected graph $\mathbf{G}$. The complement of graph $\mathbf{G}$ is also connected, choose $\mathbf{u} \in \mathbf{V}\backslash\mathbf{D_1}$, where $\mathbf{u}$ is a pendant vertex or the element of $\mathbf{V}\backslash\mathbf{D_1}$ is connected, and the set $\mathbf{D_1} = \mathbf{V}\backslash\mathbf{u}$. Clearly $\mathbf{D_1}$ dominates $\mathbf{G}$ (as $\mathbf{D} \subseteq \mathbf{D_1}$) and $\mathbf{u} \in \mathbf{V}\backslash\mathbf{D_1}$ such that $|\mathbf{N}(\mathbf{u}) \cap \mathbf{D_1}| = \mathbf{k}$. Only one thing remains to show that $< \mathbf{D_1} >$ is connected. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{D_1}$; there exists a path from joining $\mathbf{a}, \mathbf{b}, \&\mathbf{c}$ in $< \mathbf{D^1} >$ and hence $< \mathbf{D^1} >$ is connected. Thus, $\mathbf{D_1}$ is a dcfd-set in $\mathbf{G}$ with $\mathbf{p} \geq 4$. Hence $dcfd(\mathbf{G}) \leq \mathbf{p} - 1$. Let $\mathbf{D_1}'$ be a dcfd-set of $\overline{\mathbf{G}}$, and also $\overline{\mathbf{G}}$ is connected. $\mathbf{u_1} \in \mathbf{V}'\backslash\mathbf{D_{1'}}$ where $\mathbf{u_1}$ is a pendant vertex or the vertices in $\mathbf{V}'\backslash\mathbf{D_1}'$ are connected and the set $\mathbf{D_{1'}} = \mathbf{V}'\backslash\mathbf{u_1}$ clearly $\mathbf{D_1}'$ dominates $\mathbf{G}$ (as $\mathbf{D} \subseteq \mathbf{D_1}$) and $\mathbf{u_1} \in \frac{\mathbf{V}'}{\mathbf{D_{1'}}}$ therefore $|\mathbf{N}(\mathbf{u_1}) \cap \mathbf{D_1}| = \mathbf{k}$, show that $< \mathbf{D_1}' >$ is connected. Let $\mathbf{a_1}, \mathbf{b_1}, \mathbf{c_1} \in \mathbf{D_1}'$. Since $\mathbf{a_1}$, $\mathbf{b_1}$, and $\mathbf{c_1}$ are either in $\mathbf{D_1}'$ or adjacent to some vertices in $\mathbf{D}'$ and $< \mathbf{D}' >$ is connected. Hence $< \mathbf{D_1} >$ is a dcfd- set of $\overline{\mathbf{G}}$ with $\mathbf{n} \geq 4$ vertices, and therefore $dcfd(\mathbf{G}) \leq \mathbf{p} - 1$. The sum of the upper bound are $dcfd(\mathbf{G}) + dcfd(\overline{\mathbf{G}}) \leq 2\mathbf{p} - 2$, for $\mathbf{p} \geq 4$ vertices in $\mathbf{G}$ and $\overline{\mathbf{G}}$.

Remark 4.1. The above theorem is a lower bound that is sharp when $\mathbf{G} \cong \mathbf{P_p}$, where $\mathbf{P_p}$ is a path graph with $\mathbf{p} \geq 4$. The upper bound is sharp when $\mathbf{G} \cong \mathbf{T_p}$, where $\mathbf{T_p}$ is a tree graph with $\mathbf{p} \geq 4$ vertices.

Theorem 4.2. *For any connected graph* $\mathbf{G}$ *and* $\overline{\mathbf{G}}$ *with* $\mathbf{p} \geq 4$ *vertices. Then* $9 \leq dcfd(\mathbf{G}) * dcfd(\overline{\mathbf{G}}) \leq \mathbf{p}(\mathbf{p} - \mathbf{1})$

Proof:

Given $\mathbf{G}$ and $\overline{\mathbf{G}}$ are connected and also non-trivial graphs. If $\mathbf{p} = 3$, then the complement of $\overline{\mathbf{G}}$ is disconnected. Thus, the result is true for $\mathbf{p} \geq 4$ vertices in $\mathbf{G}$, and $\overline{\mathbf{G}}$ is both connected. The lower bound of the product $\mathbf{p} = 4$, then $dcfd(\mathbf{G}) = 3$, and $dcfd(\overline{\mathbf{G}}) = 3$; hence $dcfd(\mathbf{G}) * dcfd(\overline{\mathbf{G}}) = 9$; therefore, $dcfd(\mathbf{G}) * dcfd(\overline{\mathbf{G}}) \geq 9$. Let $\mathbf{p} \geq 4$, assume that $dcfd(\mathbf{G}) \geq dcfd(\overline{\mathbf{G}}) \geq 3$. Hence, $\mathbf{p} \geq 5$. By theorem-7 we know that $dcfd(\mathbf{G}) \leq \mathbf{p} - \mathbf{1} < \mathbf{p}$, and $dcfd(\overline{\mathbf{G}}) \leq \mathbf{p} - 1$. Then the upper bounds of the product if $\mathbf{p} \geq 4$, $dcfd(\mathbf{G}) * dcfd(\overline{\mathbf{G}}) \leq \mathbf{p}(\mathbf{p} - \mathbf{1})$

Remark 4.2. The above theorem is a lower bound that is sharp when $\mathbf{G} \cong \mathbf{C_p}$, where $\mathbf{C_p}$ is a cycle graph with $\mathbf{p} \geq 5$ vertices.

5. Bounds on $dcfd(\mathbf{G})$*Depending* on Various Graph Parameters

A perfect doubly connected dominating set, abbreviated as $\mathbf{pdc}$-set. A dominating set $\mathbf{D}$ is called pdc dominating set if for all vertex $\mathbf{u} \in \mathbf{V}\backslash\mathbf{D}$ such that $|\mathbf{N}(\mathbf{u}) \cap \mathbf{D}| = 1$ and the subgraphs $< \mathbf{D} >$ and $< \mathbf{V}\backslash\mathbf{D} >$ are connected. The perfect doubly connected domination number denoted by $\mathbf{pdc}(\mathbf{G})$, is the minimum cardinality of graph $\mathbf{G}$.

Theorem 5.1. *Let* $\mathbf{G}$ *be a connected graph; then* $\mathbf{pdc}(\mathbf{G}) \leq dcfd(\mathbf{G})$.

Proof:

Let $\mathbf{D}$ be a pdc dominating set of $\mathbf{G}$. Then $|\mathbf{D}| \leq \mathbf{2p} - 1$. If $\mathbf{u} \in \mathbf{V}\backslash\mathbf{D}$ and set $\mathbf{Q} = \mathbf{V}\backslash\mathbf{u}$. Clearly, $\mathbf{Q}$ dominates $\mathbf{G}$ ($\mathbf{D} \subseteq \mathbf{Q}$) and $\mathbf{u} \in \mathbf{V}\backslash\mathbf{D}$ such that $|\mathbf{N}(\mathbf{u}) \cap \mathbf{D}| = 1$, the condition holds for $\mathbf{Q}$. We have to show that $< \mathbf{Q} >$ is doubly connected. Since $\mathbf{a}, \mathbf{b} \in \mathbf{Q}$. Given that $\mathbf{a}$ and $\mathbf{b}$ are either located within $\mathbf{D}$ or are connected to certain vertices in $\mathbf{Q}$ and $\mathbf{u} \in \mathbf{V}\backslash\mathbf{D}$, where $\mathbf{u}$ is a pendant or $\mathbf{u}$ is adjacent to other vertices in $\mathbf{V}\backslash\mathbf{D}$ thus $< \mathbf{D} >$ doubly connected. Therefore, $\mathbf{Q}$ is a $\mathbf{pdc}$-set in $\mathbf{G}$; hence $\mathbf{pdc} \leq \mathbf{2p} - \mathbf{1}$. Then, by Theorem-1: $dcfd(\mathbf{G}) \leq 2\mathbf{p} - 1$ for any connected graph. Thus, we have $\mathbf{pdc} \leq dcfd$.

Remark 5.1. The above theorem sharp when $\mathbf{G} \cong \mathbf{F_p}$ where $\mathbf{F_p}$ is a friendship graph. For, $\mathbf{F_p}$ is a collection of $\mathbf{p}$ triangles with a common vertex.

Corollary 2. *Let* $\mathbf{m} \geq 2$ *and* $\mathbf{p} \geq 2$. *Then* $\mathbf{pdc}(\mathbf{K_{m,p}}) = dcfd(\mathbf{K_{m,p}}) = 2$.

7.Conclusion:

Describe the concept of a dcfd number for a graph $\mathbf{G}$ which is connected. The Nordhus-Garddum type outcomes of doubly connected fair domination. Various inequalities and bounds are observed concerning the number of vertices. A promising continuation of this work is the relationship concerning other graph parameters, between the characterization of a dcfd in a graph and its bounds.

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