RESEARCH PAPER

# MORE THEOREMS ON INTERVAL VALUED FUZZY FUNCTIONS USING STAR SHAPED ORDER

**$D.Rajan^1$ and $C.Senbagavalli^2$**
**PG & Research Department of Mathematics**
**Tranquebar Bishop Manikam Lutheran College, Porayar – 609307, Tamilnadu, South India.**
$E-mail^1$: mailto: danrajan7468@gmail.com danrajan7468@gmail.com
$E-mail^2$: mailto: senbagavalliphd@gmail.com senbagavalliphd@gmail.com

**Abstract**

Star shaped order is the basic concept as Star order. It is used to order random variables. This work explores the concept of Star shaped order within the context of fuzzy random variables and includes suitable application.

**Keywords** Star shaped order, Star shaped function, fuzzy star shaped order, Interval valued fuzzy functions, fuzzy random variables,

**How to cite this article:** D.Rajan and C.Senbagavalli. MORE THEOREMS ON INTERVAL VALUED FUZZY FUNCTIONS USING STAR SHAPED ORDER. Int J Drug Deliv Technol. 2026;16(79s):441-444. DOI: 10.25258/ijddt.16.79s.77.

## 1. Introduction

Stochastic orderings play a fundamental role in probability and statistics and have been studied by many authors. Various types of stochastic orders are currently used to compare random variables.

This work deals with the concept of the star shaped order in the context of fuzzy random variables. Fuzzy set theory was introduced by Zadeh[6], and the concept of fuzzy random variables was introduced by Kwakernaak[4,5]. In this work, we discuss the properties of the star-shaped order and provide a suitable application using fuzzy random variables.

The star shaped order is an important type of stochastic order used to compare random variables based on their variability and aging behavior. It is defined through the property that the ratio of quantile functions is increasing, which reflects a form of relative dispersion between random variables. Compared to other stochastic orders, the star shaped order lies between the usual stochastic order and the convex order, and it provides useful insights into how one random variable grows or spreads relative to another.

This work is classified into four sections. Section 1 contains the introduction, Section 2 presents the preliminaries for the study, Section 3 deals with theorems on the star-shaped order, and Section 4 provides suitable applications of the star-shaped order.

## 2. Preliminaries:

**Definition 2.1: Fuzzy set [6]**

We consider X be the universal set. Then a fuzzy set $\tilde{A} = \{(x, \mu_A(x)) / x \in X\}$ is defined by its membership function $\mu_A: X \to [0,1]$.

**Definition 2.2: $\alpha -$ cut of a fuzzy set [6]**

For each $0 \le \alpha \le 1$, the $\alpha -$cut of a set $\tilde{A}$ is denoted by $\tilde{A}_\alpha = \{ x \in X;\ \mu_{\tilde{A}}(x) \ge \alpha\}$.

**Definition2.3: Fuzzy random variable [4,5]**

X $(\omega)$ is a fuzzy random variable if and only if X $_\alpha(\omega) = [X_\alpha^L(\omega), X_\alpha^U(\omega)]$ where $X_\alpha^L(\omega)$and$X_\alpha^U(\omega)$ are both random variables for each $\alpha \in [0,1]$ and $X = \bigcup_{\alpha\in[0,1]} \alpha\, \mathrm{X}_\alpha(\omega)$.

**Definition2.4 : $\alpha -$ cut of the distribution function F of Fuzzy random variable [11] [12]**

The $\alpha -$ cut of the distribution function F of Fuzzy random variable X is defined by

$$D_\alpha = \left[Min\left(\min_{\alpha \le \beta} F(x_\beta^L), \min_{\alpha \le \beta} F(x_\beta^U)\right), Max\left(\max_{\alpha \le \beta} F(x_\beta^L), \max_{\alpha \le \beta} F(x_\beta^U)\right)\right]$$

**Definition.2.5: Stochastic order [13]**

If X and Y be fuzzy random variables with fuzzy cumulative distribution function F and G respectively then $X \le_{st} Y \Leftrightarrow F(t) \ge G(t)$ for all t.

**Definition.2.6: Star shaped order [13]**

Let X and Y be two positive fuzzy random variables such that

$E(\phi(X)) \le E(\phi(Y))$ all star shaped function $\phi: [0,\infty) \to [0,\infty)$.

provided that the expectations exist. Then X is said to be smaller Y in the star shaped order.

**Definition.2.7: Fuzzy Star shaped order**

Let X and Y be two positive fuzzy random variables such that$X \le_{SS} Y$

$$\min_{0<\alpha\le 1}(\inf E(\phi(X_\alpha^L)), \inf E(\phi(X_\alpha^U))) \le \min_{0<\alpha\le 1}(\inf E(\phi(Y_\alpha^L)), \inf E(\phi(Y_\alpha^U)))$$

$$\max_{0<\alpha\le 1}(\sup E(\phi(X_\alpha^L)), \sup E(\phi(X_\alpha^U))) \le \max_{0<\alpha\le 1}(\sup E(\phi(Y_\alpha^L)), \sup E(\phi(Y_\alpha^U)))$$

For all fuzzy star shaped function $\phi: [0,\infty) \to [0,\infty)$and $\alpha \in (0,1]$.

## 3. Theorems on interval valued fuzzy functions using star shaped orderings

**Theorem .3.1:**

Let X and Y be two positive fuzzy random variables with distribution functions F and G respectively. Then$X \le_{SS} Y$ if and only if, $\int_y^\infty x\, dF(x) \le \int_y^\infty x\, dG(x)$, $y \ge 0$.

Proof:

For a crisp random variables X and Y, the test function $\phi_y$ is defined by

$$\phi_y(x) = \begin{cases} 0, & x \le y \\ x, & x > y \end{cases}$$

is star shaped. Here $\phi_y$ is increasing in $[0,\infty)$.

Let us consider X and Y are fuzzy random variable then the fuzzy test function becomes.

$$\phi_y(x) \cong \left([\phi_y(x)]^L_\alpha, [\phi_y(x)]^U_\alpha\right)$$

$$[\phi_y(x)]^L_\alpha = \begin{cases} 0, & x \le y, \alpha \in (0,1] \\ \min_{0<\alpha\le 1}\left(\min_{0<\alpha\le 1} x^L_\alpha, \min_{0<\alpha\le 1} x^U_\alpha\right), & x > y, \alpha \in (0,1] \end{cases}$$

$$[\phi_y(x)]^U_\alpha = \begin{cases} \max_{0<\alpha\le 1}\left(\max_{0<\alpha\le 1} x^L_\alpha, \max_{0<\alpha\le 1} x^U_\alpha\right), & x \le y, \alpha \in (0,1] \\ x \quad , & x > y, \alpha \in (0,1] \end{cases}$$

both are star shaped.

We know that,

If X and Y be two positive fuzzy random variables

$$\min_{0<\alpha\le 1}\left(\inf E(\phi(X^L_\alpha)), \inf E(\phi(X^U_\alpha))\right) \le \min_{0<\alpha\le 1}\left(\inf E(\phi(Y^L_\alpha)), \inf E(\phi(Y^U_\alpha))\right)$$

$$\max_{0<\alpha\le 1}\left(\sup E(\phi(X^L_\alpha)), \sup E(\phi(X^U_\alpha))\right) \le \max_{0<\alpha\le 1}\left(\sup E(\phi(Y^L_\alpha)), \sup E(\phi(Y^U_\alpha))\right)$$

For all fuzzy star shaped function $\phi: [0,\infty) \to [0,\infty)$and $\alpha \in (0,1]$.

By the above definition we get,

$$\min_{0<\alpha\le 1}\left(\inf\int_y^\infty x^L_\alpha\, dF(x^L_\alpha), \inf\int_y^\infty x^U_\alpha\, dF(x^U_\alpha)\right) \le \min_{0<\alpha\le 1}\left(\inf\int_y^\infty x^L_\alpha\, dG(x^L_\alpha), \inf\int_y^\infty x^U_\alpha\, dG(x^U_\alpha)\right)$$

$$\max_{0<\alpha\le 1}\left(\sup\int_y^\infty x^L_\alpha\, dF(x^L_\alpha), \sup\int_y^\infty x^U_\alpha\, dF(x^U_\alpha)\right) \le \max_{0<\alpha\le 1}\left(\sup\int_y^\infty x^L_\alpha\, dG(x^L_\alpha), \sup\int_y^\infty x^U_\alpha\, dG(x^U_\alpha)\right)$$

$y \ge 0, \alpha \in (0,1]$.

Conversely,

Let $\phi$ be a star shaped function, then $h(x) = \frac{\phi(x)}{x}$ is increasing in x on $[0,\infty)$. Approximate h by a sequence of increasing step function $h_n$.

Then,

$$\min_{0<\alpha\le 1}\left(\inf\int_y^\infty x^L_\alpha\, dF(x^L_\alpha), \inf\int_y^\infty x^U_\alpha\, dF(x^U_\alpha)\right) \le \min_{0<\alpha\le 1}\left(\inf\int_y^\infty x^L_\alpha\, dG(x^L_\alpha), \inf\int_y^\infty x^U_\alpha\, dG(x^U_\alpha)\right)$$

$$\max_{0<\alpha\le 1}\left(\sup\int_y^\infty x^L_\alpha\, dF(x^L_\alpha), \sup\int_y^\infty x^U_\alpha\, dF(x^U_\alpha)\right) \le \max_{0<\alpha\le 1}\left(\sup\int_y^\infty x^L_\alpha\, dG(x^L_\alpha), \sup\int_y^\infty x^U_\alpha\, dG(x^U_\alpha)\right)$$

$$\min_{0<\alpha\le 1}\left(\inf\int_y^\infty x^L_\alpha h_n dF(x^L_\alpha), \inf\int_y^\infty x^U_\alpha h_n dF(x^U_\alpha)\right) \le \min_{0<\alpha\le 1}\left(\inf\int_y^\infty x^L_\alpha h_n dG(x^L_\alpha), \inf\int_y^\infty x^U_\alpha h_n dG(x^U_\alpha)\right)$$

$$\max_{0<\alpha\le 1}\left(\sup\int_y^\infty x^L_\alpha h_n dF(x^L_\alpha), \sup\int_y^\infty x^U_\alpha h_n dF(x^U_\alpha)\right) \le \max_{0<\alpha\le 1}\left(\sup\int_y^\infty x^L_\alpha h_n dG(x^L_\alpha), \sup\int_y^\infty x^U_\alpha h_n dG(x^U_\alpha)\right)$$

Letting $n \to \infty$, we obtain,

$$\min_{0<\alpha\le 1}\left(\inf\int_y^\infty x^L_\alpha\, dF(x^L_\alpha), \inf\int_y^\infty x^U_\alpha\, dF(x^U_\alpha)\right) \le \min_{0<\alpha\le 1}\left(\inf\int_y^\infty x^L_\alpha\, dG(x^L_\alpha), \inf\int_y^\infty x^U_\alpha\, dG(x^U_\alpha)\right)$$

$$\max_{0<\alpha\le 1}\left(\sup\int_y^\infty x^L_\alpha\, dF(x^L_\alpha), \sup\int_y^\infty x^U_\alpha\, dF(x^U_\alpha)\right) \le \max_{0<\alpha\le 1}\left(\sup\int_y^\infty x^L_\alpha\, dG(x^L_\alpha), \sup\int_y^\infty x^U_\alpha\, dG(x^U_\alpha)\right)$$

Hence proved.

**Theorem. 3.2:**

Let $\phi: [0,\infty) \to [0,\infty)$ and $g: [0,\infty) \to [0,\infty)$ be increasing and star shaped functions such that $\phi(0) = g\,(0) = 0$ then the composition of two $\phi$ and $g$ also star shaped.

**Proof:**

We have,

$\phi$ and $g$ are star shaped functions and increasing in $[0,\infty)$.

We know that,

If any star shaped functions $\phi$ and $g$ are increasing in $[0,\infty)$ then $\frac{\phi(x)}{x}$ and $\frac{g(x)}{x}$ both are increasing in x. where $x \in (0,1)$. We want to prove that the composite function of $\phi$ and $g$(ie., $\phi \circ g$) is also star shaped.

Let $\psi$ be the composite function of $\phi$ and $g$. Then

$$\psi = \phi \circ g$$

$$\psi(x) = (\phi \circ g)(x)$$

$$\psi(x) = \phi(x) \circ g(x)$$

To prove $\psi(x)$ is star shaped to prove $\frac{\psi(x)}{x}$ is increasing.

Since, the product of two increasing function is again a increasing function.

Therefore, $\frac{\psi(x)}{x}$ is increasing in $x$, $x \in (0,1)$.

Hence the composition of two star shaped function again a star shaped function .

**Theorem. 3.3.**

Let X and Y be two positive fuzzy random variables such that $X \le_{SS} Y$ and $g$ is any star star shaped function with $g(0) = 0$, then $g(X) \le_{SS} g(Y)$.

**Proof:**

We have X and Y be two positive fuzzy random variable.

We know that,

X is less than Y under star shaped order

$$\min_{0<\alpha\le 1}\left(\inf E(\phi(X^L_\alpha)), \inf E(\phi(X^U_\alpha))\right) \le \min_{0<\alpha\le 1}\left(\inf E(\phi(Y^L_\alpha)), \inf E(\phi(Y^U_\alpha))\right)$$

$$\max_{0<\alpha\le 1}\left(\sup E(\phi(X^L_\alpha)), \sup E(\phi(X^U_\alpha))\right) \le \max_{0<\alpha\le 1}\left(\sup E(\phi(Y^L_\alpha)), \sup E(\phi(Y^U_\alpha))\right)$$

For all fuzzy star shaped function $\phi: [0,\infty) \to [0,\infty)$ and $\alpha \in (0,1]$.

g is any star shaped function. Then

$$\min_{0<\alpha\le 1}(\inf E(\phi(g(X_\alpha^L))), \inf E(\phi(g(X_\alpha^U)))) \le \min_{0<\alpha\le 1}(\inf E(\phi(g(Y_\alpha^L))), \inf E(\phi(g(Y_\alpha^U))))$$

$$\max_{0<\alpha\le 1}(\sup E(\phi(g(X_\alpha^L))), \sup E(\phi(g(X_\alpha^U)))) \le \max_{0<\alpha\le 1}(\sup E(\phi(g(Y_\alpha^L))), \sup E(\phi(g(Y_\alpha^U))))$$

$$\min_{0<\alpha\le 1}\left(\inf E\left((\phi\circ g)(X_\alpha^L)\right), \inf E\left((\phi\circ g)(X_\alpha^U)\right)\right) \le \min_{0<\alpha\le 1}\left(\inf E\left((\phi\circ g)(Y_\alpha^L)\right), \inf E\left((\phi\circ g)(Y_\alpha^U)\right)\right)$$

$$\max_{0<\alpha\le 1}\left(\sup E\left((\phi\circ g)(X_\alpha^L)\right), \sup E\left((\phi\circ g)(X_\alpha^U)\right)\right) \le \max_{0<\alpha\le 1}\left(\sup E\left((\phi\circ g)(Y_\alpha^L)\right), \sup E\left((\phi\circ g)(Y_\alpha^U)\right)\right)$$

Here, we take $(\phi\circ g) = \psi$ then,

$$\min_{0<\alpha\le 1}(\inf E(\psi(X_\alpha^L)), \inf E(\psi(X_\alpha^U))) \le \min_{0<\alpha\le 1}(\inf E(\psi(Y_\alpha^L)), \inf E(\psi(Y_\alpha^U)))$$

$$\max_{0<\alpha\le 1}(\sup E(\psi(X_\alpha^L)), \sup E(\psi(X_\alpha^U))) \le \max_{0<\alpha\le 1}(\sup E(\psi(Y_\alpha^L)), \sup E(\psi(Y_\alpha^U)))$$

Since, composite function of two star shaped function.
Therefore, $\psi$ is star shaped.
From the above statement, we can conclude that

$$g(X) \le_{SS} g(Y)$$

Hence proved.

**4. Application to two stochastic processes:**

**Proposition:** Let $[X_n]$ be a sequence of independent and identically distributed random variables and let h be a star shaped function. Then the function g and g defined on the set of positive integers by $g(n) = E(h(\sum_{i=1}^{n} X_i))$ is star shaped.

**Proof :**

For given $n \ge 1$, let $A_j = [1,\dots,n+1] - \text{i}$ , j = 1, … , n+1.Then assuming h is star shaped.

Let us consider $S_n = X_1 + \cdots + X_n$ and $S_{A_j} = \frac{\sum_{i\in A_j} X_i}{X_1+\cdots+X_n}$ then,

$$\min_{0<\alpha\le 1}\left(\inf E\left(h((S_{A_j})_\alpha^L)\right), \inf E\left(h((S_{A_j})_\alpha^U)\right)\right) \le \min_{0<\alpha\le 1}\left(\inf E\left(\frac{S_{A_j}}{S_n}\, h\,(S_n)\right)_\alpha^L, \inf E\left(\frac{S_{A_j}}{S_n}\, h\,(S_n)\right)_\alpha^U\right)$$

$$\max_{0<\alpha\le 1}\left(\sup E\left(h\left((S_{A_j})_\alpha^L\right)\right), \sup E\left(h\left((S_{A_j})_\alpha^U\right)\right)\right) \le \max_{0<\alpha\le 1}\left(\sup E\left(\frac{S_{A_j}}{S_n}\, h\,(S_n)\right)_\alpha^L, \sup E\left(\frac{S_{A_j}}{S_n}\, h\,(S_n)\right)_\alpha^U\right)$$

Summing on both sides and using the assumption $X_i$'s are independent and identically distributed, we get

$$(n+1)\min_{0<\alpha\le 1}(\inf E(S_n)_\alpha^L, \inf E(S_n)_\alpha^U) \le (n)\min_{0<\alpha\le 1}\left((\inf E(S_n)_\alpha^L, \inf E(S_n)_\alpha^U\right)$$

and

$$(n+1)\max_{0<\alpha\le 1}(\sup E(S_n)_\alpha^L, \sup E(S_n)_\alpha^U) \le (n)\max_{0<\alpha\le 1}\left((\sup E(S_n)_\alpha^L, \sup E(S_n)_\alpha^U\right)$$

$$\min_{0<\alpha\le 1}\left(inf\,\frac{g_\alpha^L(n)}{\text{n}}, inf\,\frac{g_\alpha^U(n)}{\text{n}}\right) \le \min_{0<\alpha\le 1}\left(inf\,\frac{g_\alpha^L(n+1)}{\text{n}+1}, inf\,\frac{g_\alpha^U(n+1)}{\text{n}+1}\right)$$

and

$$\max_{0<\alpha\le 1}\left(sup\,\frac{g_\alpha^L(n)}{\text{n}}, sup\,\frac{g_\alpha^U(n)}{\text{n}}\right) \le \max_{0<\alpha\le 1}\left(sup\,\frac{g_\alpha^L(n+1)}{\text{n}+1}, sup\,\frac{g_\alpha^U(n+1)}{\text{n}+1}\right)$$

$n = 1,2,\dots$ and $\alpha \in (0,1]$.

which is equivalent to $g$ is star sahped.

**References:**


[1] Bartoszewicz.J (1985), Dispersive ordering and monotone failure rate distributions, Adv. Appl. Prob.17, 472 – 474.

[2] Doksum, P. J. (1969), Comparisons of Stochastic Models in Reliability Theory, Journal of Applied Probability, 6(2), 287-301.

[3] JongwooJeon, SubhashKochar, ChulGyu Park (2006), Dispersive ordering – Some applicationsa and examples, Statistical papaers 47, 227-247.

[4] Kwakernaak.H.K(1978), Fuzzy random variables – I, Definitions and theorems, Information Sciences. 15 1-29.

[5] Kwakernaak.H.K (1979), Fuzzy random variables – II, Algorithms and examples for the discretecase, Information Sciences. 17 253-278.

[6] L. A. Zadeh, Fuzzy sets, Information and Control 8(1965), 338-353.

[7] Lehmann, E. L. (1955), Ordered families of distribution. The Annals of mathematics.26,399-419.

[8] Priyakumar.J.E.L and Renganathan (2001), Stochastic orderings of fuzzy random variables, Information and Management sciences, 12 29 – 40.

[9] Puri.M.L and Ralescu.D.A (1986), Fuzzy random variables. Journal of Mathematical Analysis and Applications, 114 409-422.

[10] Rajan.D and D.Vijayabalan (2015), A note on stochastic orders, International Journal of Fuzzy mathematical archive, Vol. 9, No. 2, 2015, 245-250.

[11] Rajan.D and D.Vijayabalan (2018), Some ranking indexes of stochastic orders and their applications, Malaya Journal of Matematik, Vol. 6, No. 3, 2018, 595-609.

[12] Rajan, D., & Senthilmurugan, C. (2016). A New Approach on Up and Down Shift Proportional Fuzzy Hazard Rate Orderings of Triangular Fuzzy Random Variables. Published in International Journal of Applications of Fuzzy Sets and Artificial Intelligence, 6, 163-181.

[13] Shaked.M and Shanthikumar.J.G(2007), Stochastic orders, Sprinjer.

[14] Zimmerman.H.J, (2010) Fuzzy set theory, Wiley Interdisciplinary Reviews: computational statistics, 2 (2010) 317-332.