RESEARCH PAPER

Flow of a Viscous Incompressible Fluid between Two Parallel Plates

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Abstract:

An analysis is presented in the paper for the flow of a viscous incompressible fluid between two parallel plats, one in uniform motion and other at rest with uniform suctions. The Navier –Stoke's equations with appropriate boundary conditions reduce to second and third order non-linear differential equations. An exact solution has been obtained by using perturbation method.

Keywords: Viscous incompressible fluid, Two parallel plates, uniform suctions, Navier –Stoke's equations, Perturbation methods, suction parameters.

MSC 2010 Classification: 34D10.

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1. Introduction:

The exact solution of Navier –Stoke's equation for the generalized plane couette flow between two plates, one in uniform motion and other at rest is already known Schlichting [1]. Verma and Bansal [2] have discussed the couette flow between two parallel plates, one in uniform and the other at rest with uniform suction at the stationary plate under the assumption that the flow is due to shear and suction at the stationary wall is uniform and small. In the present problem under the same assumption the problem of generalized plane couette flow between two parallel plates, one in uniform motion and the other in rest has been discussed. The expressions for longitudinal and transverse velocities (the pressure increase in the x-direction, the shearing stress at the wall, the coefficient of skin friction and the discharge per unit breadth of the plat) have been obtained. Due to suction at the plates an adverse pressure gradient is developed which causes a back flow at a large distance from the mouth of the channel, the pressure distribution is parabolic .The coefficient of skin friction at the plates increases with the increase of suction parameter, when suction is zero, the results transform to the known generalized plane couette flow, B.R.Luthra [4], K.D.Singh and Rajesh Sharma [5].

2. Formulation:

Let us consider the two-dimensional laminar flow such that the axis of x is in between the two parallel plates and the y is measured at right angels to it, then we have

$$w = 0 \text{ and } \frac{\partial}{\partial z}(.) = 0.$$

Therefore, in this case, the equations of steady motion of viscous incompressible fluid are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial \rho}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2.1}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial \rho}{\partial x} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{2.2}$$

and
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.3}$$

u, v, w are the velocity components along x, y, z directions respectively and ν is the kinematic viscosity and ρ is the density of the fluid.

The boundary conditions are

$$y = -y_0 \;,\; \mathrm{v} = v_0 \;,\; u = 0 \;,\; y = y_0 \;,\; \mathrm{v} = -v_0 \text{ and } u = U \;. \tag{2.4}$$

$\frac{\partial v}{\partial x} = 0$ because there is uniform suction.

Therefore v is a function of y only and thus from equations (2.3)

$$\frac{\partial^2 u}{\partial x^2} = 0 ,$$

then the equations (2.1) and (2.2) become

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial \rho}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

(2.5)

$$v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial \rho}{\partial x} + \nu\frac{\partial^2 v}{\partial y^2}.$$

(2.6)

Let us introduce the following non-dimensional quantities

$$\bar{u} = \frac{u}{U}, \bar{v} = \frac{v}{v_0}, \bar{x} = \frac{x}{y_0}, \eta = \frac{y}{y_0}, \bar{p} = \frac{p}{\rho U^2}, \sigma =$$

. (2.7)

Hence, the equations (2.5),(2.6) and (2.3) take the following non-dimensional form

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\sigma}{R}\bar{v}\frac{\partial \bar{u}}{\partial \eta} = -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{1}{R}\frac{\partial^2 \bar{u}}{\partial \eta^2}$$

(2.8)

$$\bar{v}\frac{\partial \bar{v}}{\partial \eta} = -\frac{R^2}{\sigma^2}\frac{\partial \bar{p}}{\partial \eta} + \frac{1}{\sigma}\frac{\partial^2 \bar{v}}{\partial \eta^2}$$

(2.9)

and $$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\sigma}{R}\frac{\partial \bar{v}}{\partial \eta} = 0.$$

(2.10)

The boundary conditions (2.4) with the help of (2.7) become

$$\begin{aligned} &\eta = -1, \bar{v} = -1, \bar{u} = 0 \text{ and} \\ &\eta = 1, \bar{v} = 1, \bar{u} = 1 \end{aligned}.$$

(2.11)

Let

$$\bar{p}(\bar{x},\eta) = p_0 + p'(\bar{x},\eta), \bar{u}(\bar{x},\eta) = u_0 + u'(\bar{x},\eta),$$

(2.12)

where the primed quantities are the perturbations caused by the suction and P_0, u_0 are the quantities without suction (i.e. generalized plane couette flow) satisfying the equations

$$\frac{\partial p_0}{\partial \eta} = 0, \frac{\partial u_0}{\partial \bar{x}} = 0, \frac{\partial p_0}{\partial \bar{x}} = a_0 \text{ (constant)}$$

$$\text{and } u_0 = \frac{1}{2}(\eta + 1) + A(\eta^2 - 1), \text{ where } A = \frac{Ra_0}{2}$$

(2.13)

Substituting (2.12) and (2.13) in (2.8),(2.9) and (2.10), we have

$$(u_0 + u')\frac{\partial u'}{\partial \bar{x}} + \frac{\sigma}{R}v'\left(\frac{1}{2} + 2A\eta + \frac{\partial u'}{\partial \eta}\right) = -\frac{\partial p'}{\partial \bar{x}} + \frac{1}{R}\frac{\partial^2 u'}{\partial \eta^2} - a_0 + \frac{2A}{R}$$

(2.14)

$$v'\frac{\partial v'}{\partial \eta} = -\frac{R}{\sigma^2}\frac{\partial p'}{\partial \eta} + \frac{1}{\sigma}\frac{\partial^2 v'}{\partial \eta^2}$$

(2.15)

and $$\frac{\partial u'}{\partial \bar{x}} + \frac{\sigma}{R}\frac{\partial v'}{\partial \eta} = 0.$$

(2.16)

The boundary conditions (2.11) with (2.12) become

$$\begin{aligned} &\eta = -1, \bar{v} = -1, u' = 0 \text{ and} \\ &\eta = 1, v' = 1, \quad u' = 1 \end{aligned}$$

(2.17)

3. Solution of the Problem:

Let $$v' = -f(\eta).$$

(3.1)

Therefore, the equation (2.16) becomes

$$u' = \frac{\sigma}{R}\bar{x}f'(\eta) + F(\eta),$$

(3.2)

where $f'(\eta)$ and $F(\eta)$ are unknown functions of η. Substituting the values of v' and u′ from equations (3.1) and (3.2) in (2.14) and (2.15), we have

$$\frac{\partial p'}{\partial \bar{x}} = \frac{1}{R}\left[F'' + 2A - \sigma\left(\frac{1}{2}f' + \frac{1}{2}\eta f' + A\eta^2 f' - Af' - \frac{1}{2}f - 2A\eta f + f'F - fF'\right)\right]$$
$$+ \frac{\sigma}{R^2}\bar{x}\left[f''' - \sigma\left(f'^2 - ff''\right)\right] - a_0{}',$$

(3.3)

$$\frac{\partial p'}{\partial \eta} = -\frac{\sigma}{R^2}\left(\sigma ff' + f''\right)".$$

(3.4)

Now on the mouth of the channel, the pressure gradient along the axis of the channel is assumed to be same as the pressure gradient of the flow without porosity. Thus

$$\frac{\partial p_0}{\partial \bar{x}} = \frac{\partial \bar{p}}{\partial \bar{x}} = a_0 \text{ (say)}.$$

Therefore at $\bar{x} = 0, \frac{\partial p'}{\partial \bar{X}} = 0$.

Hence from equation (3.3), we have

$$F'' - \sigma\left(\frac{1}{2}f' + \frac{1}{2}\eta f' + A\eta^2 f' - Af' - \frac{1}{2}f - 2A\eta f \cdot\right.$$

. (3.5)

Differentiating (3.4) with respect to $\bar{x}$, we have

$$\frac{\partial^2 p'}{\partial \bar{x}\partial \eta} = 0.$$

(3.6)

Again differentiating (3.3) with respect to η and simplifying using the results (3.5) and (3.6), we have

$$\frac{d}{dx}\left[f''' - \sigma(f'^2 - ff'')\right] = 0$$

(3.7)

which is true for all x.

Integrating (3.7), we get

$$f''' - \sigma(f'^2 - ff'') = C,$$

(3.8)

here C is constant of integration.

The boundary conditions (2.17) are

$$\eta = -1, f(-1) = -1, f'(-1) = 0, F(-1) = 0$$
$$\text{and } \eta = 1, f(1) = -1, \quad f'(1) = 0, F(1) = 0$$

(3.9)

In order to solve (3.8), let us consider the power series for f and C near σ as

$$f = f_0 + \sigma f_1 + \sigma^2 f_2 + \sigma^3 f_3 + \ldots + \sigma^n f_n$$

(3.10)

and

$$C = C_0 + \sigma C_1 + \sigma^2 C_2 + \sigma^3 C_3 + \ldots + \sigma^n C_n$$

(3.11)

where f_i's and C_i's are independent of σ.

By substituting (3.9) and (3.10) into (3.8) and equating the coefficients of like powers of σ, we get

$$f_0''' = C_0,$$

(3.11)

$$f_1''' - f_0' + f_0 f_0'' = C_1,$$

(3.12)

$$f_2''' - 2f_0' f_1' + f_1 f_0'' + f_0 f_1'' = C_2,$$

(3.13)

etc.

In order to solve (3.11) ,(3.12) and (3.13), the boundary conditions of f_i's are

$$f_0(-1) = 1, f_n(-1) = 0 \text{ for } n \geq 1, f_n'(-1) = 0 \text{ for } n \geq 0,$$
$$f_0(-1) = -1, f_n(1) = 0 \text{ for } n \geq 1, f_n'(1) = 0 \text{ for } n \geq 0.$$

Thus the second order perturbation solution of equation (3.8) is

$$f(\eta) = \frac{1}{2}(\eta^3 - 3\eta) + \frac{\sigma}{280}(\eta^7 - 3\eta^3 + 2\eta) - \frac{\sigma^2}{1293600}(14\eta^{11} - 385\eta^9 + 198\eta^7 + 876\eta^3 - 703\eta)$$

(3.14)

and $C = 3 - \frac{81}{35}\sigma + \frac{234}{13475}\sigma^2.$

(3.15)

Again to solve the equation (3.5), let us consider the power series for F near $\sigma = 0$ as

$$F = F_0 + \sigma F_1 + \sigma^2 F_2 + \sigma^3 F_3 + \ldots + \sigma^n F_n$$

(3.16)

where F_i's are independent of σ, substituting in (3.5) the values of f and F from (3.14) and (3.16) and their derivatives, we get after equating to zero the coefficient of like powers of σ,

$$F_0'' = 0,$$

(3.17)

$$F_1'' - \left(\frac{1}{2} + \frac{1}{2}\eta + A\eta^2 - A\right)f_0' + \left(\frac{1}{2} + 2A\eta\right)f_0 - f_0' F_0 + f_0 F_0' = 0$$

(3.18)

$$F_1'' - \left(\frac{1}{2} + \frac{1}{2}\eta + A\eta^2 - A\right) f_1' + \left(\frac{1}{2} + 2A\eta\right) f_1 - f_0' F_1 -$$

(3.19)

and the boundary conditions are

$$\eta = -1, F_n(-1) = 0 \text{ for } n \geq 0 \text{ and}$$
$$\eta = 1, F_n(1) = 0 \text{ for } n \geq 0$$

(3.20)

Thus the second order perturbation solution of equation (3.5) is

$$F = \frac{\sigma}{240}\left[4A\left(\eta^6 + 45\eta^2 - 46\right) + 3\left(2\eta^5 + 5\eta^4 - 30\eta^2 - 2\eta + 25\right)\right]$$
$$-\frac{\sigma}{2822400}\begin{bmatrix} 8A\left(28\eta^{10} - 630\eta^8 - 4284\eta^6 + 1690\eta^4 - 30303\eta^2 + 291011\right) \\ +35\left(16\eta^9 + 27\eta^8 - 288\eta^6 - 144\eta^5 + 3888\eta^3 + 18756\eta^2 + 128\eta - 22419\right) \end{bmatrix}$$

(3.21)

The velocity components in the x and y directions are

$$\bar{u} = \frac{1}{2}(\eta + 1) + A(\eta^2 - 1) + \frac{\sigma \bar{x}}{R}\begin{bmatrix} \frac{3}{2}(\eta^2 - 1) + \frac{\sigma}{280}(7\eta^6 - 9\eta^2 + 2) \\ -\frac{\sigma^2}{1293600}(154\eta^{10} - 3465\eta^8 + 1386\eta^6 + 2628\eta^2 - 703) \end{bmatrix}$$
$$+\frac{\sigma}{240}\left[4A(\eta^6 + 45\eta^2 - 46) + 3(2\eta^5 + 5\eta^4 - 30\eta^2 - 2\eta + 25)\right]$$
$$-\frac{\sigma^2}{2822400}\begin{bmatrix} 8A(28\eta^{10} + 630\eta^8 - 4284\eta^6 + 16905\eta^4 - 30303\eta^2 + 291011) + \\ +35(16\eta^9 + 27\eta^8 - 288\eta^7 - 252\eta^6 - 144\eta^5 + 3888\eta^4 + 18756\eta^2 + 128\eta - 22419) \end{bmatrix}$$

(3.22)

and

$$\bar{v} = \frac{-1}{2}(\eta^3 - 3\eta) - \frac{\sigma}{280}(\eta^7 - 3\eta^3 + 2\eta) + \frac{\sigma^2}{1293600}(14\eta^{11} - 385\eta^9 + 198\eta^7 + 876\eta^3 - 703\eta)$$

.(3.23)

The pressure distribution is

$$\bar{p}(0,0) - \bar{p}(\bar{x},\eta) = \frac{\sigma}{R^2}\left(f' + \frac{3}{2}\right) - \frac{\sigma^2}{R^2}\left(\frac{1}{140} - \frac{f^2}{2}\right) - \frac{703}{1293600}\frac{\sigma^3}{R^2} - \frac{\sigma \bar{x}^2 C}{2R^2}$$

(3.24)

where C is given by (3.15).

The pressure increase in the x-direction is

$$\bar{p}(\bar{x},\eta) - \bar{p}(0,\eta) = \frac{\sigma \bar{x}^2 C}{2R^2}.$$

(3.25)

The shearing stress at the wall is

$$\tau_0 = \frac{\mu U}{y_0}\left(\frac{\partial \bar{u}}{\partial \eta}\right)_{\eta=1} = \frac{\mu U}{y_0}\left[\frac{1}{2} + 2A + \frac{2\sigma}{5}(4A - 1) + \frac{\sigma^2}{3525800}(568904A - 222005) + \frac{\sigma \bar{x}}{R}\left(3 + \frac{3\sigma}{35} + \frac{394\sigma^2}{40425}\right)\right]$$

(3.26)

The coefficient of skin friction is

$$C_f = \frac{2\tau_0}{\rho U^2} = \frac{2}{R}\left[\frac{1}{2} + 2A + \frac{2\sigma}{5}(4A - 1) + \frac{\sigma^2}{3525800}(568904A - 222005) + \frac{\sigma \bar{x}}{R}\left(3 + \frac{3\sigma}{35} + \frac{394\sigma^2}{40425}\right)\right]$$

(3.27)

The discharge per unit breadth of the plate is

$$Q = Uy_0\int_{-1}^{1} \bar{U} d\eta = 2Uy_0\left[\frac{1}{2} - A\left(\frac{2}{3} + \frac{18}{35}\sigma + \frac{44161}{80850}\sigma^2\right) - \sigma\left(\frac{\bar{x}}{R} - \frac{1}{5} - \frac{153}{800}\sigma\right)\right]$$

(3.28)

The discharge when $\sigma = 0$ is

$$Q_0 = \frac{Uy_0}{3}(3 - 4A).$$

(3.29)

which is the same as in generalized couette flow with no suction, Pai[3].

4. Conclusion:

If we draw the longitudinal velocity profile for various values of $\bar{x}$(taking $A = 1, \sigma = 1, R = 100)$, we observe that it decreases for values of $\bar{x}$ greater than -100 and for large values of $\bar{x}$, an adverse pressure gradient is developed which causes a back flow.

It is also observed that the pressure increase in the main flow is parabolic. The pressure increase with the increase of suction parameter as in a boundary layer flow of a porous plate .

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