RESEARCH PAPER

Structure of Equilibria in Finite Non-Cooperative Games

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ABSTRACT

In this study, we investigate the structure of Nash equilibrium points in finite (N)-person games. We show that, in two-player games, the interchangeability of equilibria and the convexity of the equilibrium set are equivalent properties. In contrast, this statement does not hold for three-player games. Furthermore, for completely mixed games, we derive the necessary inequality constraints on the number of pure strategies available to the players. It is proved that the equilibrium point in two-player completely mixed games is unique. However, for three-player completely mixed games, this uniqueness does not generally hold unless certain additional conditions, such as the convexity of the equilibrium set, are satisfied. For a three-player completely mixed game in which each player has two pure strategies, the equilibrium point is unique. The proof of this result requires several combinatorial arguments, which will be presented and explained in detail.

Keywords: Nash equilibrium; Completely mixed games; Equilibrium set; Convexity; Interchangeability of equilibria.

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INTRODUCTION

John Nash extended the theory of finite non-cooperative games by introducing the concept of the equilibrium point. He proved that every finite non-cooperative game possesses at least one equilibrium point in mixed strategies. In this study, we examine the impact of this theorem on the development of game theory and, more broadly, its influence on industry and other scientific disciplines. More generally, we investigate the changes that have taken place following the introduction of this theory. When the number of players is two and the game is zero-sum, the well-known von Neumann Minimax Theorem applies. In two-player zero-sum games, Kaplansky introduced the concept of completely mixed strategies in 1945 and showed that, if both players have optimal completely mixed strategies, the payoff matrix must be square and each player has an optimal strategy. In 1970, Raghavan extended this result to two-player non-zero-sum bimatrix games [1–8].

In this research, we seek to generalize the results established for two-player games to both general (N)-person games and three-player games. In three-player games, completely mixed strategies are well defined. We observe that the convexity of the equilibrium set reflects the instability of equilibrium points in bimatrix games.

It should be noted that the primary reference used in this research is the article entitled:

"Structure of Equilibria in N-person Non-cooperative Games"

by Raghavan et al. ([7]).

Nash Equilibrium

If game theory aims to provide a solution to a game, that solution must constitute an equilibrium. Suppose that game theory predicts a particular strategy profile as the solution to a game. This prediction implies that the players will choose that strategy profile. Such a prediction is valid and consistent with actual behavior only if the players indeed follow it. This occurs when each player's selected strategy is a best response to the other players' predicted strategies. These best-response strategies capture the essence of the Nash equilibrium concept. Alternatively, a Nash equilibrium is defined as a strategy profile in which no player can unilaterally change their strategy and improve their payoff. This property is fundamental, since at a Nash equilibrium no player can obtain a higher payoff by deviating unilaterally from the chosen strategy profile [8–12].

Nash Equilibrium in an *n*-Player Game

The Nash equilibrium of a static game with complete information, represented in strategic form by G, is defined as follows. A strategy profile

$$S^* = (s_1^*, s_2^*, \dots, s_n^*) \in S$$

is called a Nash equilibrium if, for every player,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*), \forall s_i \in S_i,\ \forall i \in N,$$

In other words, given the player s_{-i}^* correct belief about the opponents' choices, namely $s_i \in S_i$, player (i) should choose s_i to maximize the payoff function $u_i(s_i, s_{-i}^*)$. If a strategy s_i maximizes this function, then s_i is called a best response to $s_i \in S_i$ [13-16].

maximizes, which we denote by B_i. Then s_i is the answer to the following optimization problem: [13-16].

$$\max_{s_i \in S_i} u_i(s_i, s_{-i}^*), \forall i \in N.$$

Based on the best response function, we can say that the Nash equilibrium in a game G is achieved where we have: [13-16].

$$s_i^* \in B_i(s_{-i}^*) = \{\, s_i \in S_i \colon u_i(s_i, s_{-i}^*) \geq u_i(s_i', s_{-i}^*),\ \forall s_i' \in S_i\}.$$

And if in the set $B_i(s_{-i}^*)$ is a member of the Nash equilibrium, then the Nash equilibrium is

$$s_i^* = B_i(s_{-i}^*)$$

Focal Nash Equilibrium

The game in which players use combined testing is called an attention-motor game and is denoted by Λ(G,c), where $c =$

(c_i), $c \in \mathbb{R}^n$, is given by a set of strategies $(\eta_{-i}, \pi_{-i}, \beta_i)$, such that each player i solves the following expression: [16-18].

$$\max_{(\eta_i,\pi_i,\beta_i)} \sum_{Z_i} \eta_i\,(Z_i) E^{\,i}\,[g_i[\beta,\pi] - c_i(\pi^{\,i},\mu)]$$

subject to

$$\text{(i)}\eta_i \in \Delta(Z_i),$$
$$\text{(ii)}\beta^{\,i} \in B_{S_i},$$
$$\text{(iii)}\pi_k^{Z_i} \in \{(\pi_k^{Z_i})^*: \pi_k^{Z_i} \in \Delta(Z)\ \forall k\}, \forall Z_i. \qquad (6.1)$$

We denote the equilibrium of the game Λ(G,c) for each player i as (η, π, β), such that it solves (6.1). An equilibrium in Λ(G,c) is called pure in experiments if η is degenerate for each i. An equilibrium (η, π, β) is called pure in strategic behavior if $\beta^i(s_i)$ is the unit vector for each i and s_i. An equilibrium in Λ(G,c) is called pure if it is pure in both experiments and strategic behavior. The equilibrium represented by (η, π, β) is called a focal Nash equilibrium for Λ(G,c) if it is pure and the set of signals is equivalent to the set of actions [16–18]. That is,

$$S_i = A_i$$

and

$$\beta^{\,i} = id_{A_i}.$$

Completely Mixed Games and Equilibrium Properties in Three-Player Games

A game is said to be completely mixed if every equilibrium is completely mixed. In other words, no player can be in equilibrium with the other players while assigning zero probability to any pure strategy. In two-player zero-sum games, as well as in completely mixed non-zero-sum games, the equilibrium set consists of a single point. However, this property no longer holds for games with $N \geq 3$ players [19–20]. In this section, we construct a three-player completely mixed game whose equilibrium set ϵ consists of exactly two points. This demonstrates that, without additional assumptions, the uniqueness of equilibrium cannot be established in general.

Theorem 1: Suppose $m \geq n+l$. Then for any given $\epsilon \ni (x^0, y^0, z^0)$, there is always a (x^*, y^0, z^0) in ϵ such that $x^*$is not a complete composite.

Proof

If x^0 is not completely mixed, there is nothing to prove. Therefore, assume that x^0 is completely mixed. Without loss of generality, for (r=1,2,3),

$$K_r(x^0, y^0, z^0) = v_r = c$$

Consider the matrix

$$C = \begin{bmatrix} A & -I & 0 \\ B & 0 & -I \end{bmatrix},$$

where (I) denotes an appropriately dimensioned row vector whose entries are all equal to one.

The rank of (C) satisfies

$$n + 1 \leq m$$

Hence, there exist at least m linearly independent solutions of

$$\begin{bmatrix} A & -I & 0 \\ B & 0 & -I \end{bmatrix} \begin{bmatrix} \Pi \\ \alpha \\ \beta \end{bmatrix} = 0.$$

And clearly one of these solutions is independent of 0. Let (Π, α, β) be the solution. Clearly Π≠0; otherwise, α=β=0, and let (Π, α, β) be the normal vector. So we have

$$\begin{bmatrix} A & -I & 0 \\ B & 0 & -I \end{bmatrix} \begin{bmatrix} \Pi \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} A\Pi - \alpha \\ B\Pi - \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

that is,

$$A\Pi = \alpha\mathbf{1}, B\Pi = \beta\mathbf{1}.$$

On the other hand,

$$Ax^0 = 0, Bx^0 = 0.$$

For the x coordinate, we get and if Π is linearly independent of x^0 or $\alpha \neq 0$ or $\beta \neq 0$, then Π and x^0 are linearly independent. Now, we can create x* that is not a complete combination and $(x^*, y^0, z^0) \in \epsilon$, which completes the proof of the theorem.

Extension of Theorem 1

Remark 1: The inequality condition in Theorem 1 cannot be weakened; this is shown in the following example.

Example 1. Suppose m=3, n=2, and l=2. Suppose $K_1(i,j,k) \equiv$ (constant). We define,

$$K_2(i,j,1) = \begin{bmatrix} 2 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}, K_2(i,j,2) = \begin{bmatrix} 0 & 0 & -3 \\ -2 & -1 & 0 \end{bmatrix}.$$

where i refers to the column index and j to the row index.

Similarly,

$$K_3(i,1,k) = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix}, K_3(i,2,k) = \begin{bmatrix} 0 & -2 & -1 \\ 1 & -2 & -2 \end{bmatrix}.$$

We have $x_0 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ as an equilibrium point and $y_0 = \left(\frac{1}{2}, \frac{1}{2}\right)$, $z_0 = \left(\frac{1}{2}, \frac{1}{2}\right)$ as the only members in $s(y_0, z_0)$. Note that $m=n+l-1$ and $s(y_0, z_0)$ is a perfect combination. Of course, the game is not a perfect combination, because the strategy $(1,2,2)$ is a pure equilibrium point. Later, we will give an example with

We will construct $m=3$, $n=2$, and $l=2$, where all points of the combinatorial equilibrium are complete.

We see that the claim of Theorem 1 is clear. However, with a few additional conditions, we can prove the following.

Theorem 2. Suppose in a 3-player game $m=n+l-1$ and $(x_0, y_0, z_0) \in \epsilon$. If y_0 or z_0 is not a complete combination, then $s(y_0,z_0)$ is not a complete combination.

Proof. For $r=1,2,3$, we assume that $K_r(x_0, y_0, z_0) = v_r = 0$. We also assume that z_l^0 is zero as the last coordinate of z_0. We have,

$$C = \begin{bmatrix} A & -1 & 0 \\ B_1 & 0 & -1 \end{bmatrix},$$

where B_1 is a submatrix of B with the last row removed. Since C has $n+l-1$ rows and $n+l+1$ columns, the following equation has an axiom,

$$\begin{bmatrix} A & -1 & 0 \\ B_1 & 0 & -1 \end{bmatrix} \begin{pmatrix} \Pi \\ \alpha \\ \beta \end{pmatrix} = 0,$$

which is independent of $(x_0, 0, 0)$. Now we set Π≠0, because otherwise (Π, α, β) is an axiom vector. So we have the following cases:

1. If α≠0, and since by assumption we have for j,

$$K_2(x_0, j, z_0) = (Ax_0)_j = 0, \qquad (3.1)$$

2. If $\alpha=\beta=0$, then clearly by choosing (Π, α, β), Π and x_0 are independent;

3. If $\alpha=0$ and $\beta\neq0$, since for k<l, we have $z_k^0>0$,

$$v_3 = K_3(x_0, y_0, k) = 0,$$

And

$$(B_1x_0)_k = 0.$$

Then Π and x_0 are independent.

Let b_l be the last row of B. We have $(b_l\Pi) < \beta$ or $(b_l\Pi) \geq \beta$. Let $\tilde{x} = (1+\lambda)x_0 + \lambda\Pi$, $\hat{x} = x_0 + \lambda\Pi$ and $x^* = x_0 - \lambda\Pi$ and $x'=(1+\lambda)x0-\lambda\Pi$. We choose one of them depending on whether $\Sigma\Pi_i=0$ or $\Sigma\Pi_i=1$ and $(b_l\Pi)<\beta$ or $(b_l\Pi)\geq\beta$. So we have:

1. If $\Sigma\Pi_i = 0$ برای for positive λ, both x* and $\hat{x}$ are

combinatorial strategies but not complete combinatorial ones.

2. If $\Sigma\Pi_i = 1$ (without loss of generality, we have $\Sigma\Pi_i \neq 0$), similarly for x' and $\hat{x}$,

(a) if $\Sigma\Pi_i = 0$ and $b_l\Pi < \beta$, we choose $\hat{x}$; if $\Sigma\Pi_i = 0$ and $b_l\Pi \geq \beta$, we choose x^*;

(b) if $\Sigma\Pi_i = 1$ and $b_l\Pi < \beta$, we choose $\tilde{x}$; if $\Sigma\Pi_i = 1$ and $b_l\Pi \geq \beta$, we choose x'.

Considering cases (a) and (b) and the definitions of x^* ، $\hat{x}$ ، x', and $\tilde{x}$, we have,

$$Ax^* \leq -\lambda\alpha, A\hat{x} \leq \lambda\alpha, Ax' \leq -\lambda\alpha, A\tilde{x} \leq \lambda\alpha,$$
$$B_1x^* \leq -\lambda\beta, B_1\hat{x} \leq \lambda\beta, B_1x' \leq -\lambda\beta, B_1\tilde{x} \leq \lambda\beta,$$
$$b_lx^* \leq -\lambda\beta, b_l\hat{x} \leq \lambda\beta, b_lx' \leq -\lambda\beta, b_l\tilde{x} \leq \lambda\beta.$$

So for each j we have,

$$K_2(x_0, j, z_0) = (Ax_0)_j = 0 \Rightarrow K_2(x^*, j, z_0) = (Ax^*)_j \leq Ax^* \leq -\lambda\alpha,$$

Therefore, we can say that there exists j_0 such that,

$$K_2(x^*, j_0, z_0) = -\lambda\alpha.$$

In fact, when $y_{j_0} > 0$, the equilibrium condition occurs. Similarly, when we have k<l,

$$K_3(x_0, y_0, k) = (B_1x_0)_k \Rightarrow K_3(x^*, y_0, k) = (B_1x^*)_k \leq B_1x^* \leq -\lambda\beta,$$

Which occurs for k_0<l, so,

$$K_3(x^*, y_0, k) = (B_1x^*)_k = -\lambda\beta.$$

Combining these relations shows that $(x^*, y_0, z_0) \in \epsilon$. Similarly, the other states can be eliminated. Therefore, $s(y_0, z_0)$ is not a complete combination.

Remark 2. When m=n+l-2 and when player 2 or 3 lose a pure strategy in an equilibrium (x_0, y_0, z_0), we cannot conclude that $s(y_0, z_0)$ is a complete combination. Even when m=n=l=2, we have simple examples.

Unique Analysis of Equilibrium Points

We begin this section by proving the following result.

Theorem 3. Suppose that in a three-player game, the set of combinatorial equilibria is complete and convex. Then ϵ has only one member

Proof. Let $K_r(x, y, z)$ be the expected payoff of the r-th player, when x, y, and z are the combined strategies chosen by players 1, 2, and 3. By the converse proof, let ϵ have two equilibrium points (x^0, y^0, z^0) and (x', y', z'). By the convexity assumption, for 0≤λ≤1, we have

$$(\lambda x^0 + (1-\lambda)x',\ \lambda y^0 + (1-\lambda)y',\ \lambda z^0 + (1-\lambda)z') \in \epsilon.$$

Since the game is a perfect combination, the following states occur.

1. By definition, for every pure strategy i, we have

$$K_1(i, \lambda y^0 + (1-\lambda)y', \lambda z^0 + (1-\lambda)z') = c_1,$$

That is,

$$\begin{aligned} &K_1(i, \lambda y^0 + (1-\lambda)y', \lambda z^0 + (1-\lambda)z') \\ &= \lambda^2 K_1(i, y^0, z^0) + (1-\lambda)^2 K_1(i, y', z') \\ &+\lambda(1-\lambda)(K_1(i, y^0, z') + K_1(i, y', z^0)) = c_1. \end{aligned}$$

Since,

$$K_1(i, y^0, z^0) = v_1^0, K_1(i, y', z') = v_1',$$

So we have

$$K_1(i, y^0, z') + K_1(i, y', z^0) = \alpha_1.$$

2. By definition, for every pure strategy j, we have

$$K_2(\lambda x^0 + (1-\lambda)x', j, \lambda z^0 + (1-\lambda)z') = c_2,$$

That is,

$$\begin{aligned} &K_2(\lambda x^0 + (1-\lambda)x', j, \lambda z^0 + (1-\lambda)z') \\ &= \lambda^2 K_2(x^0, j, z^0) + (1-\lambda)^2 K_2(x', j, z') \\ &+\lambda(1-\lambda)(K_2(x^0, j, z') + K_2(x', j, z^0)) = c_2. \end{aligned}$$

Since,

$$K_2(x^0, j, z^0) = v_2^0, K_2(x', j, z') = v_2',$$

So we have,

$$K_2(x^0, j, z') + K_2(x', j, z^0) = \alpha_2.$$

3. By definition, for every pure strategy we have k

$$K_3(\lambda x^0 + (1-\lambda)x', \lambda y^0 + (1-\lambda)y', k) = c_3,$$

That is,

$$\begin{aligned} &K_3(\lambda x^0 + (1-\lambda)x', \lambda y^0 + (1-\lambda)y', k) \\ &= \lambda^2 K_3(x^0, y^0, k) + (1-\lambda)^2 K_3(x', y', k) \\ &+\lambda(1-\lambda)(K_3(x^0, y', k) + K_3(x', y^0, k)) = c_3. \end{aligned}$$

Since,

$$K_3(x^0, y^0, k) = v_3^0, K_3(x', y', k) = v_3',$$

So we have,

$$K_3(x^0, y', k) + K_3(x', y^0, k) = \alpha_3.$$

For t>0 and small t, we set

$$x^* = (1+t)x^0 - tx', y^* = (1+t)y^0 - ty', z^* = (1+t)z^0 - tz'.$$

where x*, y*, and z* are all mixed strategies. Since

$$(x^0, y^0, z^0) \neq (x', y', z'),$$

We have t>0 such that at least one of them is not a complete combination. We show

$$(x^*, y^*, z^*) \in \epsilon.$$

According to the above situations, we have:

1.
$$\begin{aligned} K_1(i, y^*, z^*) &= (1+t)^2 K_1(i, y^0, z^0) + t^2 K_1(i, y', z') \\ &-t(1+t)(K_1(i, y^0, z') + K_1(i, y', z^0)) \\ &\equiv (1+t)^2 v_1^0 + t^2 v_1' - t(1+t)\alpha_1. \end{aligned}$$

2.
$$\begin{aligned} K_2(x^*, j, z^*) &= (1+t)^2 K_2(x^0, j, z^0) + t^2 K_2(x', j, z') \\ &-t(1+t)(K_2(x^0, j, z') + K_2(x', j, z^0)) \\ &\equiv (1+t)^2 v_2^0 + t^2 v_2' - t(1+t)\alpha_2. \end{aligned}$$

3.
$$\begin{aligned} K_3(x^*, y^*, k) &= (1+t)^2 K_3(x^0, y^0, k) + t^2 K_3(x', y', k) \\ &-t(1+t)(K_3(x^0, y', k) + K_3(x', y^0, k)) \\ &\equiv (1+t)^2 v_3^0 + t^2 v_3' - t(1+t)\alpha_3. \end{aligned}$$

As a result $(x^*, y^*, z^*) \in \epsilon$, which contradicts the assumption of complete composition of ϵ. So this theorem is complete. □

Remark 3. Let m=3, n=2, and k=2 to construct a counterexample for the uniqueness of equilibrium points in complete combinatorial games. When m = n = k = 2, we have the following result.

Theorem 4 (Main Theorem). Suppose a 3-player game has pure strategies for each player. If the game is complete

combinatorial, then the equilibrium point is unique.
In the following, we must state some preliminary points to prove this theorem. As in previous chapters, we assume,
$a_1 = K_1(1,1,1) - K_1(2,1,1), b_1 = K_2(1,1,1) - K_2(1,2,1),$
$a_2 = K_1(1,1,2) - K_1(2,1,2), b_2 = K_2(1,1,2) - K_2(1,2,2),$
$a_3 = K_1(1,2,1) - K_1(2,2,1), b_3 = K_2(2,1,1) - K_2(2,2,1),$
$a_4 = K_1(1,2,2) - K_1(2,2,2), b_4 = K_2(2,1,2) - K_2(2,2,2),$
$c_1 = K_3(1,1,1) - K_3(1,1,2),$
$c_2 = K_3(1,2,1) - K_3(1,2,2),$
$c_3 = K_3(2,1,1) - K_3(2,1,2),$
$c_4 = K_3(2,2,1) - K_3(2,2,2).$
Let (p,(1-p)), (q,(1-q)) and (r,(1-r)) be the combined strategies for 3 players. For the sake of simplicity, we denote the combined strategies by (p,q,r).
Let $(p_0,q_0,r_0) \in \epsilon$ and $0 < p_0,q_0,r_0 < 1$. Therefore, we have,

$$q_0r_0(a_1 - a_2 - a_3 - a_4) + q_0(a_2 - a_4) + r_0(a_3 - a_4) + a_4 = 0,$$
$$p_0r_0(b_1 - b_2 - b_3 - b_4) + p_0(b_2 - b_4) + r_0(b_3 - b_4) + b_4 = 0,$$
$$p_0q_0(c_1 - c_2 - c_3 - c_4) + p_0(c_2 - c_4) + q_0(c_3 - c_4) + c_4 = 0.$$

As a result,

$$\lambda_1 = a_1 - a_2 - a_3 + a_4, \lambda_2 = a_2 - a_4, \lambda_3 = a_3 - a_4, \lambda_4 = a_4,$$
$$\mu_1 = b_1 - b_2 - b_3 - b_4, \mu_2 = b_2 - b_4, \mu_3 = b_3 - b_4, \mu_4 = b_4,$$
$$v_1 = c_1 - c_2 - c_3 - c_4, v_2 = c_2 - c_4, v_3 = c_3 - c_4, v_4 = c_4.$$

Lemma 1. When *m*=*n*=*k*=2 the game is complete with (p_0,q_0,r_0)∈ϵ and p_0 and q_0 are unique for r_0; in other words, if $(p^*,q^*,r_0) \in \epsilon$ then p$(p_0,q_0,r_0) \in \epsilon$ $p^* = p_0$ and $q^* = q_0$. The same claim holds for p_0 and q_0.
Proof. We consider the following equations,

$$\begin{aligned} qr\lambda_1 + q\lambda_2 + r\lambda_3 + \lambda_4 &= 0, \\ pr\mu_1 + p\mu_2 + r\mu_3 + \mu_4 &= 0, \qquad (1)) \\ pqv_1 + pv_2 + qv_3 + v_4 &= 0. \end{aligned}$$

Let us assume that (p_0,q_0,r_0) is a solution of the above system, and that every (p,q,r) holds true in ϵ with the condition $0 \le p,q,r \le 1$. Clearly, q_0 v_1+v_2, r_0 $\lambda_1+\lambda_2$, and r_0 $\mu_1+\mu_2$ cannot all be zero, so $(1,q_0,r_0)$ that holds true in the above system of equations contradicts the completeness of the game. Therefore, at least one of them is not zero. Let us assume that r_0 $\lambda_1+\lambda_2 \neq 0$, then we have,

$$q_0(r_0\lambda_1 + \lambda_2) + (r_0\lambda_3 + \lambda_4) = 0,$$
$$q^*(r_0\lambda_1 + \lambda_2) + (r_0\lambda_3 + \lambda_4) = 0,$$

As a result, q*=q_0 and

$$(p^*,q_0,r_0),(p_0,q_0,r_0) \in \epsilon,$$

So p*=p=. Otherwise, ϵ contains a line segment connecting (p^*,q_0,r_0) and (p_0,q_0,r_0), which contradicts the completeness of ϵ.
Similarly, if we have $r_0\mu_1 + \mu_2 \neq 0$,

$$q_0(r_0\mu_1 + \mu_2) + (r_0\mu_3 + \mu_4) = 0,$$
$$q^*(r_0\mu_1 + \mu_2) + (r_0\mu_3 + \mu_4) = 0,$$

As a result, q^*=q_0 and

$$(p^*,q_0,r_0),(p_0,q_0,r_0) \in \epsilon,$$

So $p^* = p_0$. Next, we show that the occurrence of the two relations $r_0\lambda_1 + \lambda_2 = 0$ and $r_0\mu_1 + \mu_2 = 0$ is not possible. In the case where $q_0v_1 + v_2 \neq 0$, from the third equation of the system (1), we have,

$$p_0(q_0v_1 + v_2) = -(q_0v_3 + v_4) \Rightarrow p_0 = -\frac{(q_0v_3+v_4)}{(q_0v_1+v_2)},$$

We will arrange,

$$p = -\frac{(qv_3+v_4)}{(qv_1+v_2)},$$

p is a continuous differentiable function in a neighborhood Δ of q_0 with the condition 0<p<1 when q∈Δ. By Deriving p with respect to q, we have,

$$\frac{dp}{dq} = \frac{v_1v_4 - v_2v_3}{(qv_1 + v_2)^2}, \text{ for } q \in \Delta.$$

If $v_1v_4 - v_2v_3 = 0$ then p≡p_0 for q∈Δ. That is, the line segment $\{(p_0,q,r_0): q \in \Delta\} \subseteq \epsilon$ which is impossible.
If $v_1v_4 - v_2v_3 > 0$ then $\frac{dp}{dq} > 0$ and so in this case the function $p = -\frac{(qv_3+v_4)}{(qv_1+v_2)}$ is an increasing function of q in Δ.
We put,

$$p_1 = \sup\{p: (p,q,r_0) \in \epsilon\}.$$

Clearly, there exists q_1 such that $(p_1,q_1,r_0) \in \epsilon$. Also p_1<1 and q_1<1. Since $r_0\lambda_1 + \lambda_2 = r_0\mu_1 + \mu_2 = 0$ then $q_1v_1 + v_2 \neq 0$. This shows that $\frac{dp}{dq}|_{q_1} > 0$ and hence $(p_2,q_2,r_0) \in \epsilon$ exists with the condition $p_2 > p_1$which contradicts the maximal condition p_1. Similarly, we get a contradiction for when $v_1v_4 - v_2v_3 < 0$. And this completes the proof of the lemma.
Lemma 2. Suppose ϵ is a complete composite and for each *r* we have,

$$f(r) = (r\lambda_3 + \lambda_4)(r\mu_3 + \mu_4)v_1 - (r\lambda_3 + \lambda_4)(r\mu_1 + \mu_2)v_2 - (r\lambda_1 + \lambda_2)(r\mu_3 + \mu_4)v_3 + (r\lambda_1 + \lambda_2)(r\mu_1 + \mu_2)v_4 \equiv 0.$$

Then ϵ has a unique point.
Proof. Suppose that (p_0,q_0,r_0), $(p^*,q^*,r^*)$are two distinct points in ϵ if possible. By Lemma 1, $p_0 \neq p^*$, $q_0 \neq q^*$, and $r_0 \neq r^*$. In the following analysis, the following cases occur:
(a)) $r_0\lambda_1 + \lambda_2 = 0$. $r_0\mu_1 + \mu_2 = 0$. Clearly $(p^*,q^*,r_0) \in \epsilon$, and by Lemma 1, $p^* = p_0$ and $q^* = q_0$ must be true, which contradicts our assumption.
(b) For every $(p_1,q_1,r_1) \in \epsilon$, we have $r_1\lambda_1 + \lambda_2 \neq 0$, $r_1\mu_1 + \mu_2 \neq 0$. We assume that $\epsilon_3 = \{r_1: (p_1,q_1,r_1) \in \epsilon\}$ is closed on [0,1]. So, by the converse argument, we will show that ϵ_3 is open. We assume that $r_1 \in \epsilon_3$. There exists an open interval Δ_{r_1} such that,

$$p(r) = -\frac{(r\mu_3+\mu_4)}{r\mu_1 + \mu_2}, q(r) = -\frac{(r\lambda_3+\lambda_4)}{r\lambda_1 + \lambda_2},$$

That for every $r \in \Delta_{r_1}$, $0 < p(r),q(r) < 1$.
Also, (p(r),q(r),r) holds in the first and second equations (1). In addition, the assumption f(r)≡0 and any $r \in \Delta_{r_1}$implies,

$$(p(r)q(r)v_1 + p(r)v_2 + q(r)v_3 + v_4)(r\lambda_1 + \lambda_2)(r\mu_1 + \mu_2) \equiv 0.$$

Also $r\lambda_1 + \lambda_2 \neq 0$, $r\mu_1 + \mu_2 \neq 0$. So p(r) and q(r) in the third equation (1) hold. Consequently, for every $r \in \Delta_{r_1}$, (p(r),q(r),r)∈ϵ; that is, ϵ_3 is open. Consequently $\epsilon_3 = [0,1]$ and $\epsilon_3 \neq \emptyset$. This contradicts the complete combinability of

ϵ.

(C)

$$r_0\lambda_1 + \lambda_2 \neq 0, r_0\mu_1 + \mu_2 = 0$$
$$r^*\lambda_1 + \lambda_2 \neq 0, r^*\mu_1 + \mu_2 = 0,$$

Because $r_0 \neq r^*$ and $\mu_1 = \mu_2 = 0$. Also $\mu_3=\mu_4=0$. Such a game cannot be a perfect combination.

(D)

$$r_0\lambda_1 + \lambda_2 = 0, r_0\mu_1 + \mu_2 \neq 0$$
$$r^*\lambda_1 + \lambda_2 = 0, r^*\mu_1 + \mu_2 \neq 0,$$

Which is similar to (C).

(E)

$$r_0\lambda_1 + \lambda_2 \neq 0, r_0\mu_1 + \mu_2 = 0$$
$$r^*\lambda_1 + \lambda_2 = 0, r^*\mu_1 + \mu_2 \neq 0,$$

where $\mu_1\neq0$ otherwise $\mu_2=0$ which contradicts the assumption $r^*\ \mu_1+\mu_2\neq0$. Similarly $\mu_2,\mu_3,\mu_4\neq0$ (so $r_0\ \mu_3+\mu_4=0$, $r^*\ \mu_3+\mu_4\neq0$). Furthermore (μ_1,μ_2) and (μ_3,μ_4) are linearly independent since $r_0\mu_1+\mu_2=0$ and $r_0\mu_3+\mu_4=0$. So for $\alpha\neq0$ we have $(\mu_3,\mu_4)=\alpha(\mu_1,\mu_2)$, and,

$$p^* = -\frac{r^*\mu_3 + \mu_4}{r^*\mu_1 + \mu_2} = -\alpha.$$

We place in the deleted neighborhood Δ_{r_0} of the point r_0,

$$p(r) = -\frac{r\mu_3 + \mu_4}{r\mu_1 + \mu_2}$$
$$q(r) = -\frac{r\lambda_3 + \lambda_4}{r\lambda_1 + \lambda_2}.$$

Since $\mu_1, \lambda_1 \neq 0$ and $r_0\mu_1 + \mu_2 = 0$, p(r) and q(r) are defined in this omitted neighborhood. Furthermore, according to our assumption on f(r) for every $r\in\Delta(r_0)$ except $r=r_0$, $(p(r),q(r),r)$ holds in the three equations of system (1). Also, $p(r)=p^*$ at all these points, which contradicts Lemma 1.

(C)

$$r_0\lambda_1 + \lambda_2 \neq 0, r_0\mu_1 + \mu_2 = 0$$
$$r^*\lambda_1 + \lambda_2 \neq 0, r^*\mu_1 + \mu_2 \neq 0.$$

As before $\mu_1, \mu_2, \mu_3, \mu_4 \neq 0$. Also, for each $r\neq r_0$, we have,

$$p^* = -\frac{r\mu_3 + \mu_4}{r\mu_1 + \mu_2},$$

Which is similar to case (e).

The above cases complete the proof of the lemma.

Remark 4. If $(p_0, q_0, r_0) \in \epsilon$ then $f(r_0)=0$.

Proof of Theorem 4. We consider the following matrix,

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix},$$

which is related to the purely combinatorial game. Since this matrix lacks pure equilibrium, we have at least one negative coordinate for each of the following vectors,

$$v_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, v_2 = \begin{pmatrix} a_2 \\ b_2 \\ -c_1 \end{pmatrix}, v_3 = \begin{pmatrix} a_3 \\ -b_1 \\ c_2 \end{pmatrix}, v_4 = \begin{pmatrix} a_4 \\ -b_2 \\ -c_2 \end{pmatrix},$$
$$v_5 = \begin{pmatrix} -a_1 \\ b_3 \\ c_3 \end{pmatrix}, v_6 = \begin{pmatrix} -a_2 \\ b_4 \\ -c_3 \end{pmatrix}, v_7 = \begin{pmatrix} -a_3 \\ -b_3 \\ c_4 \end{pmatrix}, v_8 = \begin{pmatrix} -a_4 \\ -b_4 \\ -c_4 \end{pmatrix}.$$

We assume that the matrix A has no zero element, which means that there will be no equilibrium when two players use pure strategies. For an arbitrary matrix A, the following 6. Conditions can hold,

$$S_2: b_3b_4 < 0, \text{ و } c_3c_4 < 0,$$
$$S_3: a_1a_2 < 0, \text{ و } c_1c_3 < 0,$$
$$S_4: a_3a_4 < 0, \text{ و } c_2c_4 < 0,$$
$$S_5: a_1a_3 < 0, \text{ و } b_1b_3 < 0,$$
$$S_6: a_2a_4 < 0, \text{ و } b_2b_4 < 0.$$

It is entirely possible that none of these hold.

We consider the following equations,

$$qr\lambda_1 + q\lambda_2 + r\lambda_3 + \lambda_4 = 0,$$
$$pr\mu_1 + p\mu_2 + r\mu_3 + \mu_4 = 0, \qquad (2)$$
$$pqv_1 + pv_2 + qv_3 + v_4 = 0.$$

We assume that we have $(p_0, q_0, r_0) \in \epsilon$.

$q_0(r_0\lambda_1 + \lambda_2) + (r_0\lambda_3 + \lambda_4) = 0 \rightarrow (1)$,

$p_0(r_0\mu_1 + \mu_2) + (r_0\mu_3 + \mu_4) = 0 \rightarrow (2)$,

$(r_0\lambda_1 + \lambda_2)(r_0\mu_1 + \mu_2)(p_0q_0v_1 + p_0v_2 + q_0v_3 + v_4) = 0$.

That is,

$$p_0(r_0\mu_1 + \mu_2)q_0(r_0\lambda_1 + \lambda_2)v_1 + p_0(r_0\lambda_1 + \lambda_2)(r_0\mu_1 + \mu_2)v_2 + q_0(r_0\mu_1 + \mu_2)(r_0\lambda_1 + \lambda_2)v_3 + (r_0\lambda_1 + \lambda_2)(r_0\mu_1 + \mu_2)v_4 = 0.$$

Using (1) and (2), we can rewrite the above relationship as follows,

$$(r_0\mu_3 + \mu_4)(r_0\lambda_3 + \lambda_4)v_1 - (r_0\mu_3 + \mu_4)(r_0\lambda_1 + \lambda_2)v_2 - (r_0\lambda_3 + \lambda_4)(r_0\mu_1 + \mu_2)v_3 + (r_0\lambda_1 + \lambda_2)(r_0\mu_1 + \mu_2)v_4 = 0,$$

Therefore, the function,

$$f(r) = (r\mu_3 + \mu_4)(r\lambda_3 + \lambda_4)v_1 - (r\mu_3 + \mu_4)(r\lambda_1 + \lambda_2)v_2 - (r\lambda_3 + \lambda_4)(r\mu_1 + \mu_2)v_3 + (r\lambda_1 + \lambda_2)(r\mu_1 + \mu_2)v_4 = 0,$$

has one root at $r=r_0$. By Lemma 2, we assume that $f(r)\not\equiv0$ and that f(r) is either linear or quadratic. If it is linear (the obvious case), then for every $(p_0, q_0, r_0) \in \epsilon$, r_0 is unique, and by Lemma 1, we have a unique equilibrium point. The non-obvious case occurs when f(r) is quadratic. Without loss of generality, we assume there are three quadratics, which we call f_p, f_q, and f_r. We have,

$$f_p(0) = b_4c_4a_1 - b_3c_4a_2 - b_4c_3a_3 + b_3c_3a_4,$$
$$f_p(1) = b_2c_2a_1 - b_1c_2a_2 - b_2c_1a_3 + b_1c_1a_4,$$
$$f_q(0) = a_4c_4b_1 - a_3c_4b_2 - a_4c_2b_3 + a_3c_2b_4,$$
$$f_q(1) = a_2c_3b_1 - a_1c_3b_2 - a_2c_1b_3 + a_1c_1b_4,$$
$$f_r(0) = a_4b_4c_1 - a_2b_4c_2 - a_4b_2c_3 + a_2b_2c_4,$$
$$f_r(1) = a_3b_3c_1 - a_1b_3c_2 - a_3b_1c_3 + a_1b_1c_4.$$

We can also calculate the following expressions, which we will need later,

$$f_p \boxed{,} \left(\frac{c_3}{c_3 - c_1}\right) = (c_1c_4 - c_2c_3)f_q(1), \text{ when } c_1c_3 < 0,$$
$$f_p \boxed{,} \left(\frac{b_3}{b_3 - b_1}\right) = (b_1b_4 - b_2b_3)f_r(1), \text{ when } b_1b_3 < 0,$$
$$f_p \boxed{,} \left(\frac{c_4}{c_4 - c_2}\right) = (c_1c_4 - c_2c_3)f_q(0), \text{ when } c_2c_4 < 0,$$
$$f_p \boxed{,} \left(\frac{b_4}{b_4 - b_2}\right) = (b_1b_4 - b_2b_3)f_r(0), \text{ when } b_2b_3 < 0,$$
$$f_q \boxed{,} \left(\frac{c_2}{c_2 - c_1}\right) = (c_1c_4 - c_2c_3)f_p(1), \text{ when } c_1c_2 < 0,$$
$$f_q \boxed{,} \left(\frac{c_4}{c_4 - c_3}\right) = (c_1c_4 - c_2c_3)f_p(0), \text{ when } c_3c_4 < 0,$$
$$f_q \boxed{,} \left(\frac{a_3}{a_3 - a_1}\right) = (a_1a_4 - a_2a_3)f_r(1), \text{ when } a_1a_3 < 0,$$
$$f_q \boxed{,} \left(\frac{a_4}{a_4 - a_2}\right) = (a_1a_4 - a_2a_3)f_r(0), \text{ when } a_2a_4 < 0,$$

$$f_r \boxed{,} \left(\frac{a_4}{a_4 - a_3}\right) = (a_1a_4 - a_2a_3)f_q(0), \text{ when } a_3a_4 < 0,$$
$$f_r \boxed{,} \left(\frac{b_4}{b_4 - b_3}\right) = (b_1b_4 - b_2b_3)f_p(0), \text{ when } b_3b_4 < 0,$$
$$f_r \boxed{,} \left(\frac{a_2}{a_2 - a_1}\right) = (a_1a_4 - a_2a_3)f_q(1), \text{ when } a_1a_2 < 0,$$
$$f_r \boxed{,} \left(\frac{b_2}{b_2 - b_1}\right) = (b_1b_4 - b_2b_3)f_p(1), \text{ when } b_1b_2 < 0.$$

Two matrices are called equivalent if they have the same nonzero entries and the same sign structure. There are 212 distinct matrix modulus equivalents. Of these, certain matrices may be purely balanced and can therefore be eliminated; for example, if the first column

$$\begin{pmatrix} + \\ + \\ + \end{pmatrix}$$

We can remove it. For certain matrices, we can directly check that $f_p(0)f_p(1) \le 0$ or $f_q(0)f_q(1) \le 0$ or $f_r(0)f_r(1) \le 0$, then there is only one positive root for a quadratic, and by Lemma 1, the equilibrium is unique. For example, if the symbolic structure is as follows,

For example, if the symbol structure is as follows,

$$\begin{pmatrix} -,+,-,+ \\ +,+,-,- \\ +,-,+,- \end{pmatrix}, f_r(0)f_r(1) < 0.$$

We have considered all the cases, and the proof of the theorem depends on the location of the roots in a suitable quadratic. We will now discuss some of the common cases that are repeated in all other cases:

1. First case: None of the conditions S_1, S_2, ⋯ , or S_6 hold. For example, consider the following case,

$$A = \begin{pmatrix} -,+,-,+ \\ +,+,-,- \\ +,-,+,- \end{pmatrix},$$

We know that $f_r(0) = a_4b_4c_1 - a_2b_4c_2 - a_4b_2c_3 + a_2b_2c_4$ and $f_r(1) = a_3b_3c_1 - a_1b_3c_2 - a_3b_1c_3 + a_1b_1c_4$ So by substituting the elements of matrix A, we have,

$$f_r(0) < 0, \text{ و } f_r(1) > 0.$$

Therefore $f_r(0)f_r(1) < 0$ and there is a unique root for r-quadratic in the interval (0,1). Consequently, by Lemma 1, the equilibrium point is unique.

Remark 5. In fact, when none of the conditions S_1 to S_6 hold, and no element of A is zero, then we can check that each row has exactly two positive signs and two negative signs, when ϵ is a complete composite. We can also claim that $a_1a_4<0$, $b_1b_4<0$, $a_2a_3<0$, $c_1c_3<0$, $b_2b_3<0$ and $c_2c_3<0$.

2. Second case: We assume that one of the conditions S_1, S_2, ⋯ , or S_6 holds. Then we must: $a_4a_2<0$ and $b_4b_2<0$. For example,

$$A = \begin{pmatrix} -,-,-,+ \\ -,+,-,- \\ -,-,+,- \end{pmatrix},$$

Considering the sign of the elements of matrix A, we have,

$$0 < \frac{a_4}{a_4 - a_2}, \frac{b_4}{b_4 - b_2} < 1.$$

We assume,

$$p = \frac{b_4}{b_4 - b_2}, q = \frac{a_4}{a_4 - a_2}, r = 0.$$

We have,

$$K_1(1, q, 2) = K_1(2, q, 2),$$
$$K_2(p, 1, 2) = K_2(p, 2, 2),$$
$$K_3(p, 1, 2) < K_3(p, q, 1).$$

We know that ϵ is a complete combination; that is,

$$K_3(p, q, 1) - K_3(p, q, 2) = \frac{a_4b_4c_1 - a_2b_4c_2 - a_4b_2c_3 + a_2b_2c_4}{(a_4 - a_2)(b_4 - b_2)}.$$

We know that,

$$K_3(p, q, 1) > K_3(p, q, 2) \Rightarrow K_3(p, q, 1) - K_3(p, q, 2) > 0,$$

On the other hand, we know that $a_2a_4<0$ and $b_2b_4<0$, so

$$(a_4 - a_2)(b_4 - b_2) < 0,$$

And so we must have,

$$a_4b_4c_1 - a_2b_4c_2 - a_4b_2c_3 + a_2b_2c_4 = f_r(0) < 0.$$

Also fr (1)<0; because otherwise there is only one root in the interval (0,1) for r-quadratic and the proof for that has already been discussed. Since we have $a_1a_4-a_2a_3<0$ and $a_2a_4<0$,

$$f_q\left(\frac{a_4}{a_4 - a_2}\right) = (a_1a_4 - a_2a_3)f_r(0) > 0.$$

In the first row with the symbolic structure [-,-,-,+] of the equation,

$$r_0(q_0a_1 + (1 - q_0)a_3) + (1 - r_0)(q_0a_2 + (1 - q_0)a_4) = 0,$$

In the first row with the symbolic structure [-,-,-,+] of the equation,

$$q_0a_2 + (1 - q_0)a_4 > 0 \Rightarrow q_0(a_2 - a_4) > -a_4 \Rightarrow q_0(a_4 - a_2) < a_4,$$

That is, for every (p,q,r)∈ϵ we have $q_0 < \frac{a_4}{a_4 - a_2}$, and since $a_4 - a_2 > 0$ then

$$q < \frac{a_4}{a_4 - a_2}.$$

We know that there is only one acceptable root for the second-degree q in the interval $\left(0, \frac{a_4}{a_4 - a_2}\right)$ because $f_q\left(\frac{a_4}{a_4 - a_2}\right) > 0$; so if $f_q(0) < 0$, ϵ has a unique member.

Suppose we have $f_q(0) > 0$and $f_q(1) > 0$;

$$f_p\left(\frac{c_3}{c_3 - c_1}\right) = (c_1c_4 - c_2c_3)f_q(1) > 0,$$

And

$$f_p\left(\frac{b_4}{b_4 - b_2}\right) = (b_1b_4 - b_2b_3)f_r(0) < 0,$$

For every $(p_0, q_0, r_0) \in \epsilon$, we have,

$$\frac{b_4}{b_4 - b_2} < p_0 < \frac{c_3}{c_3 - c_1},$$

And p_0 is the unique positive root for the quadratics in this interval. Therefore, ϵ has a unique member.

3. Third case: Suppose that two of the following cases hold: S_1, S_2, or ⋯ or S_6. We set,

$$A = \begin{pmatrix} +,+,-,+ \\ +,-,-,+ \\ -,+,+,- \end{pmatrix},$$

where S_1 and S_2 hold, as we already know,

$$\left(1, \frac{c_2}{c_2 - c_1}, \frac{b_2}{b_2 - b_1}\right) \notin \epsilon, \text{و} \left(0, \frac{c_4}{c_4 - c_3}, \frac{b_4}{b_4 - b_3}\right) \notin \epsilon,$$

That is,

$$\frac{f_p(1)}{(b_2 - b_1)(c_2 - c_1)} = \frac{b_2c_2a_1 - b_1c_2a_2 - b_2c_1a_3 + b_1c_1a_4}{(b_2 - b_1)(c_2 - c_1)} < 0,$$

And

$$\frac{f_p(0)}{(b_4-b_3)(c_4-c_3)} = \frac{b_4c_4a_1 - b_3c_4a_2 - b_4c_3a_3 + b_3c_3a_4}{(b_4-b_3)(c_4-c_3)} > 0.$$

Consequently $f_p(1)f_p(0) < 0$ and so ϵ has a unique member.

On the other hand, we deal with all other matrices in a similar way. When A has only nonzero entries, the proof of Theorem 4 is complete.

Remark 6. If four of the conditions $S_1,\cdots,S_6$ hold and if A has all nonzero entries, then the game has a pure equilibrium. We have observed this from the computer output.

Further remarks on N-player games

Some of the theorems proved in the previous sections can be extended to general N-player games as follows.

Theorem 5 Let $m_1,\ldots,m_N$ (N≥3) be the number of pure strategies for a player in a noncooperative N-player game. Also suppose that for each i

$$m_i \geq \sum_{i\neq j} m_j - (N-3).$$

Then, for each equilibrium

$$(x_1^0,\ldots,x_i^0,\ldots,x_N^0) \in \epsilon$$

We can balance again.

$$\left(x_1^0,\ldots,x_{i-1}^0,x_i,x_{i+1}^0,\ldots,x_N^0\right)$$

Find where x_i is not a complete combination.

Theorem 6. In the above theorem, we assume that for each *i,* we have

$$m_i = \sum_{i\neq j} m_j - (N-2).$$

If in equilibrium

$$\left(x_1^0,\ldots,x_i^0,\ldots,x_N^0\right)$$

If one of the x_i is not a complete combination for i≠j, then we can find another equilibrium.

$$\left(x_1^0,\ldots,x_{i-1}^0,x_i,x_{i+1}^0,\ldots,x_N^0\right)$$

Find x_i where x_i is also not a complete combination.

Theorem 7 Suppose we have an N-player game that is a complete combination. If the equilibrium set is convex, then this set contains only one element.

References

[1] Abdoli, G. (2012). Game Theory and Its Applications: Games with Incomplete Information, Evolutionary Games, and Cooperative Games. Academic Center for Education, Culture and Research (ACECR), University of Tehran. (In Persian).

[2] Amann M. and Sorensen I., (1991). "The energy and sulphur emission database of the RAINS model", International Institute for Applied Systems, 46 pp.

[3] KAPLANSKY, I. A contribution to von-Neumann's theory of games, Ann. Math., 46, 1945.

[4] NASH, J. F.: Non-cooperative games, Ann. Math., 54, 1951.

[5] PARTHASARATHY, T., and T. E. S. RAGHAVAN; *Some Topics in Two Person Games* 1971. Elsevier Publishing Company, New York.

[6] RAGHAVAN, T. E. S.: Completely mixed strategies in bimatrix games, J. London Math. Soc. (2), 2, 1970.

[7] Chin, H. H., Parthasarathy, T., Raghavan, T. E. S. (1974). Structure of equilibria in N-person non-cooperative games. *International Journal of Game Theory*, 3, 1–19.

[8] Harsanyi, J. C. (1968). Games with incomplete information played by "Bayesian" players part II. Bayesian equilibrium points. *Management Science*, 14(5), 320–334.

[9] Harsanyi, J. C., Selten, R. (1972). A Generalized Nash Solution for two-person Bargaining Games with incomplete information. *Management Science*, 18(5-part-2), 80–106.

[10] Klaassen, G. (1993). Trade-offs in emission trading. *Economic Instruments for Air Pollution Control*, Kluwer Academic Publishers.

[11] McDonald, I. M. and Solow, R. M. (1981). Wage bargaining and employment. *American Economic Review*, 71, 896–908.

[12] Mehimann, A. (1988). *Applied differential games*. Plenum Press, NY, London.

[13] Nash Jr., J. F. (1950). The bargaining problem. *Econometrica: Journal of the Econometric Society*, 155–162.

[14] Nash, J. (1953). Two-person co-operative games. *Econometrica*, 21, 128–140.

[15] Newbery, D. M. (1990). Acid rain. *Economic Policy*, 5(11), 297–346.

[16] Raiffa, H. (1953). Arbitration schemes for generalized two-person games. *Annals of Mathematics Studies*, 28, 361–387.

[17] Roth, A. E. (1979). *Lecture Notes in Economics and Mathematical Systems*.

[18] Shogren, J. F., Crocker, T. D. (1991). Cooperative and noncooperative protection against transferable and filterable externalities. *Environmental and Resource Economics*, 1(2), 195–214.

[19] Tahvonen, O., Kaitala, V., Pohjola, M. (1993). A Finnish-Soviet acid rain game: noncooperative equilibria, cost efficiency, and sulfur agreements. *Journal of Environmental Economics and Management*, 24(1), 87–100.

[20] Vernon, J. L. (1989). Market impacts of sulphur control: the consequences for coal. London: IEA Coal Research.