

Bayesian Estimation of Hazard and Reverse Hazard Rates for the three parameters Gumbel Distribution under LINEX loss (LLF) function

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Abstract

The Gumbel distribution is an important probability model in extreme-value theory and reliability analysis. In the present study, the hazard rate and reverse hazard rate functions of the three-parameter Gumbel distribution are investigated within a Bayesian framework. Approximate Bayes estimators are obtained using Lindley's approximation under the LINEX loss function. The proposed estimation procedure incorporates prior information about the unknown parameters and provides an alternative to the classical Maximum Likelihood Estimation method. A simulation study is conducted for different sample sizes to assess the performance of the proposed estimators. The performance is evaluated in terms of mean estimates and Mean Squared Errors. The results indicate that the Approximate Bayes Estimators generally provide estimates closer to the true parameter values with lower Mean Squared Errors than the corresponding Maximum Likelihood Estimators. Furthermore, the Mean Squared Errors generally decrease as the sample size increases. The findings demonstrate the usefulness of the Bayesian Lindley approximation approach for estimating the hazard and reverse hazard rate functions of the three-parameter Gumbel distribution under an asymmetric loss function.

Keywords: Gumbel Distribution; Bayesian Estimation; Hazard Rate; Reverse Hazard Rate; LINEX Loss Function; Lindley's Approximation; Maximum Likelihood Estimation.

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1. Introduction

Reliability analysis plays an important role in the study of lifetime and failure-time data, where the estimation of various reliability characteristics provides useful information about the behaviour of a system or unit over time. Among the important characteristics of a lifetime distribution, the hazard rate function describes the instantaneous failure risk at a specified time conditional on survival up to that time. The reverse hazard rate function, on the other hand, is associated with the ratio of the probability density function to the cumulative distribution function and provides a complementary measure for studying the behaviour of a lifetime distribution. Thus, the joint consideration of hazard and reverse hazard rate functions can provide a more comprehensive understanding of the underlying lifetime model.

The Gumbel distribution is one of the important distributions in extreme-value theory and has been widely considered for modelling extreme observations. The flexibility of the three-parameter form makes it useful for statistical modelling when location, scale, and an additional shape-related parameter are required. The theoretical development and applications of extreme-value distributions have been discussed extensively in the literature (Gumbel, 1958; Kotz and Nadarajah, 2000; Coles, 2001).

Bayesian methods provide a useful framework for parameter and reliability estimation by combining

the information from the observed sample with prior information about the unknown parameters. Bayesian estimation of reliability characteristics has been studied for several lifetime distributions under different loss functions. For example, Srivastava and Yadav (2018) investigated the Bayesian approximation of the reliability function of a three-parameter generalized compound Rayleigh distribution under the squared error loss function. In another study, Srivastava and Yadav (2018) considered the Bayesian approximation of the hazard rate function of the same distribution under the general entropy loss function. These studies demonstrate the applicability of Bayesian approximation methods for the estimation of reliability characteristics under different loss functions.

The choice of an appropriate loss function is an important aspect of Bayesian estimation because different loss functions impose different penalties on estimation errors. In particular, the LINEX loss function is asymmetric and allows unequal penalties for overestimation and underestimation. Zellner (1986) developed Bayesian estimation and prediction under asymmetric loss functions. Subsequently, Srivastava and Yadav (2024) considered reliability estimation under LINEX loss for the Weibull model using Bayesian Lindley approximation, further illustrating the usefulness of an asymmetric loss function in reliability estimation.

In the context of the Gumbel distribution, Pandey and Srivastava (2025) studied the approximation of the three parameters of the Gumbel model through Lindley Bayes. They further investigated the reliability estimation of the three-parameter Gumbel distribution using Lindley's approximation under the Bayesian framework. These studies provide a basis for further investigating other important reliability characteristics of the three-parameter Gumbel distribution within the Bayesian framework.

For Bayesian estimation, the posterior expectations involved in the estimation of functions of unknown parameters may lead to complicated integral expressions that are difficult to evaluate analytically. Lindley (1980) proposed an approximate Bayesian method for evaluating such posterior expectations, while Tierney and Kadane (1986) also developed accurate approximations for posterior moments and marginal densities. In the present study, Lindley's approximation is employed to obtain approximate Bayesian estimates of the reliability characteristics considered.

Motivated by the above developments, the present study considers the estimation of two important reliability characteristics, namely the **hazard rate function and the reverse hazard rate function**, for the three-parameter Gumbel distribution. A Bayesian estimation procedure is developed under the **LINEX loss function**, and Lindley's approximation is employed to obtain the corresponding Approximate Bayes Estimators. The resulting estimates are compared with the corresponding Maximum Likelihood Estimates through a simulation study for different sample sizes. The performance of the estimators is assessed using mean estimates and Mean Squared Error (MSE).

The main contribution of this study is therefore to provide a unified investigation of the hazard and reverse hazard rate functions of the three-parameter Gumbel distribution under an asymmetric Bayesian loss framework. The study extends the existing Bayesian reliability work on the Gumbel model by considering both hazard-related characteristics and examining their estimation performance under the LINEX loss function using Lindley's approximation. Let be a random variable following the three-parameter Gumbel distribution with parameters α, β and γ . The probability density function (pdf) and cumulative distribution function (cdf) of the distribution are given by

$$f(t) = \beta e^{-\beta(t-\gamma-\alpha)} e^{-e^{-\beta(t-\gamma-\alpha)}} \quad (1.1)$$

the cdf of three parameter Gumbel distribution is

$$F(t) = e^{-e^{-\beta(t-\gamma-\alpha)}}$$

The above distribution serves as the common underlying model for the estimation of both the hazard rate and reverse hazard rate functions considered in this study. The corresponding reliability function is obtained from the cdf as

$$R(t) = 1 - F(t) = 1 - e^{-e^{-\beta(t-\gamma-\alpha)}} \quad (1.2)$$

The hazard rate function represents the instantaneous risk of failure at time t , conditional on survival up to that time. For the three-parameter Gumbel distribution, the hazard rate function is obtained by substituting the above pdf and reliability function into its definition.

$$H(t) = \frac{f(t)}{R(t)} = \frac{\beta e^{-\beta(t-\gamma-\alpha)} e^{-e^{-\beta(t-\gamma-\alpha)}}}{1 - e^{-e^{-\beta(t-\gamma-\alpha)}}} \quad (1.3)$$

$$\text{Let, } u = t - \gamma - \alpha$$

Then

$$H(t) = \frac{\beta e^{-\beta u} e^{-e^{-\beta u}}}{1 - e^{-e^{-\beta u}}} \quad y = e^{-\beta u}$$

$$\log H(t) = \log \left[\frac{\beta e^{-\beta u} e^{-e^{-\beta u}}}{1 - e^{-e^{-\beta u}}} \right] \quad (1.4)$$

$$\log H(t) = \log \beta - \beta u - e^{-\beta u} - \log(1 - e^{-e^{-\beta u}}) \quad (1.5)$$

Then

$$H = \frac{\beta y e^{-y}}{1 - e^{-y}}$$

Expression of β in terms of H

$$H(1 - e^{-y}) = \beta y e^{-y}$$

Hence,

$$\beta = \frac{H(1 - e^{-y})}{y e^{-y}} \quad (1.6)$$

$$y = e^{-\beta u}$$

$$\text{Derivative } \frac{d\beta}{dH}$$

Differentiating w.r.t H :

$$\frac{d\beta}{dH} = \frac{1 - e^{-y}}{y e^{-y}}$$

Substituting back $y = e^{-\beta u}$:

$$\frac{d\beta}{dH} = \frac{1 - e^{-e^{-\beta(t-\gamma-\alpha)}}}{e^{-\beta(t-\gamma-\alpha)} e^{-e^{-\beta(t-\gamma-\alpha)}}} \quad (1.7)$$

$$h(\beta | x) = \frac{\prod_{i=1}^n [\beta e^{-(\beta(x-\gamma-\alpha))} e^{-e^{-(\beta(x-\gamma-\alpha))}}] \frac{b^a}{\Gamma(a)} \beta^{a-1} e^{-\beta b}}{\int_0^\infty \prod_{i=1}^n [\beta e^{-(\beta(x-\gamma-\alpha))} e^{-e^{-(\beta(x-\gamma-\alpha))}}] \frac{b^a}{\Gamma(a)} \beta^{a-1} e^{-\beta b} d\beta} \quad (1.8)$$

Represents the posterior distribution of the parameter β . It is obtained by combining the likelihood function of the three-parameter Gumbel distribution with a gamma prior for β .

Here,

• $\prod_{i=1}^n \beta e^{-(\beta x_i)} e^{-e^{-(\beta(x-\gamma-\alpha))}}$ is the likelihood

• $G(\beta | x) = \frac{b^a}{\Gamma(a)} \beta^{a-1} e^{-\beta b}$ is the Gamma prior for β

and the denominator is the normalizing constant that ensures the posterior integrates to 1.

Thus, $h(\beta | x)$ represents the Bayesian posterior of β based on the observed sample x .

$$h(\beta | t) = \frac{\beta^{n+a-1} e^{-\beta(b+\sum u_i)} \prod_{i=1}^n e^{-e^{-\beta u_i}}}{\int_0^\infty \beta^{n+a-1} e^{-\beta(b+\sum u_i)} \prod_{i=1}^n e^{-e^{-\beta u_i}} d\beta}$$

$$h(\beta | t) = \frac{\beta^{n+a-1} e^{-\beta(b+\sum u_i)} e^{-\sum e^{-\beta u_i}}}{\int_0^\infty \beta^{n+a-1} e^{-\beta(b+\sum u_i)} e^{-\sum e^{-\beta u_i}} d\beta} \quad (1.9) \quad j_3(\beta | \gamma) = \frac{1}{\Gamma_\xi} \gamma^{-\xi} \beta^{\xi+1} e^{\left[-\frac{\beta}{\gamma}\right]} \quad (1.18)$$

Using the appropriate transformation and asymptotic approximation, the posterior distribution can be expressed in the following form:

$$h(\beta | t) \propto \beta^n e^{-\beta(b+\sum u_i)} \prod_{i=1}^n e^{-e^{-\beta u_i}} \cdot \beta^{a-1} e^{-\beta b}$$

$$h(\beta | t) = \frac{(b+\sum u_i)^{n+a}}{\Gamma(n+a)} \beta^{n+a-1} e^{-\beta(b+\sum u_i)}$$

(1.10)

now we replace $\beta = H$

$$h(H | t) = \frac{(b+\sum u_i)^{n+a}}{\Gamma(n+a)} H^{n+a-1} e^{-H(b+\sum u_i)}$$

(1.11)

$$h(H | t) = \frac{(b+\sum u_i)^{n+a}}{\Gamma(n+a)} \left(\frac{\beta e^{-\beta u} e^{-e^{-\beta u}}}{1 - e^{-e^{-\beta u}}} \right)^{n+a-1} e^{-\left(\frac{\beta e^{-\beta u} e^{-e^{-\beta u}}}{1 - e^{-e^{-\beta u}}} \right)(b+\sum u_i)}$$

(1.12)

where, $u = t - \gamma - \alpha$, $u_i = x_i - \gamma - \alpha$

this is the posterior distribution of Hazard rate function.

$$h(H | t) = \frac{(b+\sum u_i)^{n+a}}{\Gamma(n+a)} H^{n+a-1} e^{-H(b+\sum u_i)}$$

(1.13)

Thus, the posterior distribution of the unknown parameters associated with the hazard rate function is obtained. This posterior distribution forms the basis for subsequent Bayesian estimation.

Bayes estimator under LINEX (Linear Exponential) loss function for the Hazard rate function.

The Bayes estimate of the hazard rate function under the LINEX loss function is derived using the posterior distribution of the unknown parameters.

$$\hat{H}_{BL} = -\frac{1}{k} [E_h(e^{-kH})]$$

(1.14)

$$\hat{H}_{BL} = -\frac{1}{k} \log E(e^{-kH} | t)$$

$$E(e^{-kH}) = \int_0^\infty e^{-kH} h(H | t) dH$$

$$= \int_0^\infty e^{-kH} \frac{(b+\sum u_i)^{n+a}}{\Gamma(n+a)} H^{n+a-1} e^{-H(b+\sum u_i)} dH$$

$$= \frac{(b+\sum u_i)^{n+a}}{\Gamma(n+a)} \int_0^\infty e^{-kH} H^{n+a-1} e^{-H(b+\sum u_i)} dH$$

$$\int_0^\infty e^{-kH} H^{n+a-1} e^{-H(b+\sum u_i)} dH =$$

$$\int_0^\infty H^{n+a-1} e^{-(k+(b+\sum u_i))H} dH \quad (1.15)$$

After solving the equation (1.14) we get,

$$\hat{H}_{BL} = \frac{n+a}{k} \log \left(1 + \frac{k}{b+\sum u_i} \right)$$

The Bayesian approach to estimating the unknown parameters α, β and γ

The combined prior distribution corresponding to the parameters α, β , and γ can be expressed as

$$J(\alpha, \beta, \gamma) = j_1(\alpha) j_2(\gamma) j_3(\beta | \gamma)$$

Taking

$$j_1(\alpha) = c$$

(1.16)

$$j_2(\gamma) = \frac{1}{\delta} e^{-\frac{\gamma}{\delta}}$$

(1.17)

Joint posterior with likelihood equation number (1.5) and (1.19) we get;

$$h^*(\alpha, \beta, \gamma) = \frac{\gamma^{-\xi} \beta^{\xi+1} e^{\left[-\left(\frac{\gamma+\beta}{\gamma}\right)\right]} L(\underline{x} | \alpha, \beta, \gamma)}{\iiint \gamma^{-\xi} \beta^{\xi+1} e^{\left[-\left(\frac{\gamma+\beta}{\gamma}\right)\right]} L(\underline{x} | \alpha, \beta, \gamma) d\alpha d\beta d\gamma} \quad (1.20)$$

The approximate Bayes estimators

$$V(\Theta) = v(\alpha, \beta, \gamma)$$

$$\hat{v}_{AB} = E(v | \underline{x}) = \frac{\iiint v(\alpha, \beta, \gamma) J^*(\alpha, \beta, \gamma) d\alpha d\beta d\gamma}{\iiint J^*(\alpha, \beta, \gamma) d\alpha d\beta d\gamma} \quad (1.21)$$

• Lindley Approximation

Lindley's approximation provides an efficient analytical alternative to exact Bayesian computation by approximating posterior expectations around the maximum likelihood estimates. This approach avoids complex multidimensional integrations while retaining the essential Bayesian structure.

$$E(v(\alpha, \beta, \gamma | \underline{x})) = V(\Theta) + \frac{1}{2} [S(v_{11}\sigma_{11} + v_{12}\sigma_{12} + v_{13}\sigma_{13}) + [T(v_{21}\sigma_{21} + v_{22}\sigma_{22} + v_{23}\sigma_{23}) + A(v_{31}\sigma_{31} + v_{32}\sigma_{32} + v_{33}\sigma_{33}) + v_1 a_1 + v_2 a_2 + v_3 a_3 + a_4 + a_5] + O\left(\frac{1}{n^2}\right)] \quad (1.22)$$

Evaluated at MLE = $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ where;

$$a_1 = \rho_1 \sigma_{11} + \rho_2 \sigma_{12} + \rho_3 \sigma_{13} \quad (1.23)$$

$$a_2 = \rho_1 \sigma_{21} + \rho_2 \sigma_{22} + \rho_3 \sigma_{23}$$

(1.24)

$$a_3 = \rho_1 \sigma_{31} + \rho_2 \sigma_{32} + \rho_3 \sigma_{33}$$

(1.25)

$$a_4 = v_{12} \sigma_{12} + v_{13} \sigma_{13} + v_{23} \sigma_{23}$$

(1.26)

$$a_5 = \frac{1}{2} (v_{11} \sigma_{11} + v_{22} \sigma_{22} + v_{33} \sigma_{33}) \quad (1.27)$$

$$S = [\sigma_{11} \ell_{111} + 2\sigma_{12} \ell_{121} + 2\sigma_{13} \ell_{131} + 2\sigma_{23} \ell_{231} + \sigma_{22} \ell_{221} + \sigma_{33} \ell_{331}] \quad (1.28)$$

$$T = [\sigma_{11} \ell_{112} + 2\sigma_{12} \ell_{122} + 2\sigma_{13} \ell_{132} + 2\sigma_{23} \ell_{232} + \sigma_{22} \ell_{222} + \sigma_{33} \ell_{332}] \quad (1.29)$$

$$A = [\sigma_{11} \ell_{113} + 2\sigma_{12} \ell_{123} + 2\sigma_{13} \ell_{133} + 2\sigma_{23} \ell_{233} + \sigma_{22} \ell_{223} + \sigma_{33} \ell_{333}] \quad (1.30)$$

For applying Lindley approximation to equation number (1.22)

The Likelihood function described by equation number (1.5)

$$\log H(t) = \ell = \log \beta - \beta u - e^{-\beta u} - \log(1 - e^{-e^{-\beta u}})$$

where, $q = e^{-\beta u}$

$$\ell = \log \beta - \beta u - q - \log(1 - e^{-q})$$

$$\ell_1 = \frac{\partial \ell}{\partial \alpha} = \beta - \beta e^{-\beta u} + \frac{\beta e^{-\beta u} e^{-e^{-\beta u}}}{1 - e^{-e^{-\beta u}}}$$

$$\begin{aligned}
 &= \beta - \beta q + \frac{\beta q e^{-q}}{1-e^{-q}} \\
 \ell_2 = \frac{\partial \ell}{\partial \beta} &= \frac{1}{\beta} - u + u e^{-\beta u} - \frac{u e^{-\beta u} e^{-\beta u}}{1-e^{-\beta u}} \\
 &= \frac{1}{\beta} - u + u q - \frac{u q e^{-q}}{1-e^{-q}} \\
 \ell_3 = \frac{\partial \ell}{\partial \gamma} &= \beta - \beta e^{-\beta u} + \frac{\beta e^{-\beta u} e^{-\beta u}}{1-e^{-\beta u}} \\
 &= \beta - \beta q + \frac{\beta q e^{-q}}{1-e^{-q}} \\
 \ell_{11} &= -\beta^2 e^{-\beta u} + \beta^2 e^{-2\beta u} + \frac{\beta^2 e^{-\beta u} e^{-\beta u} (1-e^{-\beta u})}{1-e^{-\beta u}} - \\
 &\frac{\beta^2 e^{-2\beta u} e^{-2\beta u}}{(1-e^{-\beta u})^2} \\
 &= -\beta^2 q + \beta^2 q^2 + \frac{\beta^2 q e^{-q} (1-q)}{1-e^{-q}} - \frac{\beta^2 q^2 e^{-2q}}{(1-e^{-q})^2} \\
 \ell_{22} &= -\frac{1}{\beta^2} + u^2 e^{-\beta u} - u^2 e^{-2\beta u} + \\
 &\frac{u^2 e^{-\beta u} e^{-\beta u} (1-e^{-\beta u})}{1-e^{-\beta u}} - \frac{u^2 e^{-2\beta u} e^{-2\beta u}}{(1-e^{-\beta u})^2} \\
 &= -\frac{1}{\beta^2} + u^2 q - u^2 q^2 + \frac{u^2 q e^{-q} (1-q)}{1-e^{-q}} - \frac{u^2 q^2 e^{-2q}}{(1-e^{-q})^2} \\
 \ell_{33} &= -\beta^2 e^{-\beta u} + \beta^2 e^{-2\beta u} + \frac{\beta^2 e^{-\beta u} e^{-\beta u} (1-e^{-\beta u})}{1-e^{-\beta u}} - \\
 &\frac{\beta^2 e^{-2\beta u} e^{-2\beta u}}{(1-e^{-\beta u})^2} \\
 &= -\beta^2 q + \beta^2 q^2 + \frac{\beta^2 q e^{-q} (1-q)}{1-e^{-q}} - \frac{\beta^2 q^2 e^{-2q}}{(1-e^{-q})^2} \\
 \ell_{12} &= -1 + e^{-\beta u} + \beta u e^{-\beta u} - \frac{u e^{-\beta u} e^{-\beta u}}{1-e^{-\beta u}} + \\
 &\frac{\beta u e^{-\beta u} e^{-\beta u} (1-e^{-\beta u})}{1-e^{-\beta u}} - \frac{\beta u e^{-2\beta u} e^{-2\beta u}}{(1-e^{-\beta u})^2} \\
 &= -1 + q + \beta u q - \frac{u q e^{-q}}{1-e^{-q}} + \frac{\beta u q e^{-q} (1-q)}{1-e^{-q}} - \frac{\beta u q^2 e^{-2q}}{(1-e^{-q})^2} \\
 \ell_{21} &= -1 + e^{-\beta u} + \beta u e^{-\beta u} - \frac{u e^{-\beta u} e^{-\beta u}}{1-e^{-\beta u}} + \\
 &\frac{\beta u e^{-\beta u} e^{-\beta u} (1-e^{-\beta u})}{1-e^{-\beta u}} - \frac{\beta u e^{-2\beta u} e^{-2\beta u}}{(1-e^{-\beta u})^2} \\
 &= -1 + q + \beta u q - \frac{u q e^{-q}}{1-e^{-q}} + \frac{\beta u q e^{-q} (1-q)}{1-e^{-q}} - \frac{\beta u q^2 e^{-2q}}{(1-e^{-q})^2} \\
 \ell_{23} &= -1 + e^{-\beta u} + \beta u e^{-\beta u} - \frac{u e^{-\beta u} e^{-\beta u}}{1-e^{-\beta u}} + \\
 &\frac{\beta u e^{-\beta u} e^{-\beta u} (1-e^{-\beta u})}{1-e^{-\beta u}} - \frac{\beta u e^{-2\beta u} e^{-2\beta u}}{(1-e^{-\beta u})^2} \\
 &= -1 + q + \beta u q - \frac{u q e^{-q}}{1-e^{-q}} + \frac{\beta u q e^{-q} (1-q)}{1-e^{-q}} - \frac{\beta u q^2 e^{-2q}}{(1-e^{-q})^2} \\
 \ell_{13} &= -\beta^2 e^{-\beta u} + \beta^2 e^{-2\beta u} + \frac{\beta^2 e^{-\beta u} e^{-\beta u} (1-e^{-\beta u})}{1-e^{-\beta u}} - \\
 &\frac{\beta^2 e^{-2\beta u} e^{-2\beta u}}{(1-e^{-\beta u})^2} \\
 &= -\beta^2 q + \beta^2 q^2 + \frac{\beta^2 q e^{-q} (1-q)}{1-e^{-q}} - \frac{\beta^2 q^2 e^{-2q}}{(1-e^{-q})^2}
 \end{aligned}$$

$$\begin{aligned}
 \ell_{31} &= -\beta^2 e^{-\beta u} + \beta^2 e^{-2\beta u} + \frac{\beta^2 e^{-\beta u} e^{-\beta u} (1-e^{-\beta u})}{1-e^{-\beta u}} - \\
 &\frac{\beta^2 e^{-2\beta u} e^{-2\beta u}}{(1-e^{-\beta u})^2} \\
 &= -\beta^2 q + \beta^2 q^2 + \frac{\beta^2 q e^{-q} (1-q)}{1-e^{-q}} - \frac{\beta^2 q^2 e^{-2q}}{(1-e^{-q})^2} \\
 \ell_{32} &= -1 + e^{-\beta u} + \beta u e^{-\beta u} - \frac{u e^{-\beta u} e^{-\beta u}}{1-e^{-\beta u}} + \\
 &\frac{\beta u e^{-\beta u} e^{-\beta u} (1-e^{-\beta u})}{1-e^{-\beta u}} - \frac{\beta u e^{-2\beta u} e^{-2\beta u}}{(1-e^{-\beta u})^2} \\
 &= -1 + q + \beta u q - \frac{u q e^{-q}}{1-e^{-q}} + \frac{\beta u q e^{-q} (1-q)}{1-e^{-q}} - \frac{\beta u q^2 e^{-2q}}{(1-e^{-q})^2}
 \end{aligned}$$

Where, $q = e^{-\beta u}$

$$\ell_{111} = \beta^3 q - 3\beta^3 q^2 + 2\beta^3 q^3 + \frac{\beta^3 q e^{-q} (1-3q+2q^2)}{1-e^{-q}} - \frac{3\beta^3 q^2 e^{-2q} (1-q)}{(1-e^{-q})^2} + \frac{2\beta^3 q^3 e^{-3q}}{(1-e^{-q})^3} \quad (1.31)$$

$$\ell_{222} = \frac{2}{\beta^3} + u^3 q - 3u^3 q^2 + 2u^3 q^3 + \frac{u^3 q e^{-q} (1-3q+2q^2)}{1-e^{-q}} - \frac{3u^3 q^2 e^{-2q} (1-q)}{(1-e^{-q})^2} + \frac{2u^3 q^3 e^{-3q}}{(1-e^{-q})^3} \quad (1.32)$$

$$\ell_{333} = \beta^3 q - 3\beta^3 q^2 + 2\beta^3 q^3 + \frac{\beta^3 q e^{-q} (1-3q+2q^2)}{1-e^{-q}} - \frac{3\beta^3 q^2 e^{-2q} (1-q)}{(1-e^{-q})^2} + \frac{2\beta^3 q^3 e^{-3q}}{(1-e^{-q})^3} \quad (1.33)$$

$$\ell_{112} = \beta^2 u q - 2\beta^2 u q^2 + \frac{\beta^2 u q e^{-q} (1-2q)}{1-e^{-q}} - \frac{2\beta^2 u q^2 e^{-2q}}{(1-e^{-q})^2}$$

$$\ell_{212} = \beta^2 u q - 2\beta^2 u q^2 + \frac{\beta^2 u q e^{-q} (1-2q)}{1-e^{-q}} - \frac{2\beta^2 u q^2 e^{-2q}}{(1-e^{-q})^2}$$

$$\ell_{213} = \beta^2 u q - 2\beta^2 u q^2 + \frac{\beta^2 u q e^{-q} (1-2q)}{1-e^{-q}} - \frac{2\beta^2 u q^2 e^{-2q}}{(1-e^{-q})^2}$$

$$\ell_{312} = \beta^2 u q - 2\beta^2 u q^2 + \frac{\beta^2 u q e^{-q} (1-2q)}{1-e^{-q}} - \frac{2\beta^2 u q^2 e^{-2q}}{(1-e^{-q})^2}$$

$$\ell_{121} = \ell_{211} = \ell_{112} = \ell_{332} = \ell_{323} = \ell_{233} = \ell_{223} = \ell_{123} = \ell_{132} = \ell_{213} = \ell_{231} = \ell_{321} \quad (1.34)$$

$$\ell_{311} = \beta^3 q - 3\beta^3 q^2 + 2\beta^3 q^3 + \frac{\beta^3 q e^{-q} (1-3q+2q^2)}{1-e^{-q}} - \frac{3\beta^3 q^2 e^{-2q} (1-q)}{(1-e^{-q})^2} + \frac{2\beta^3 q^3 e^{-3q}}{(1-e^{-q})^3}$$

$$\ell_{313} = \beta^3 q - 3\beta^3 q^2 + 2\beta^3 q^3 + \frac{\beta^3 q e^{-q} (1-3q+2q^2)}{1-e^{-q}} - \frac{3\beta^3 q^2 e^{-2q} (1-q)}{(1-e^{-q})^2} + \frac{2\beta^3 q^3 e^{-3q}}{(1-e^{-q})^3}$$

$$\ell_{333} = \beta^3 q - 3\beta^3 q^2 + 2\beta^3 q^3 + \frac{\beta^3 q e^{-q} (1-3q+2q^2)}{1-e^{-q}} - \frac{3\beta^3 q^2 e^{-2q} (1-q)}{(1-e^{-q})^2} + \frac{2\beta^3 q^3 e^{-3q}}{(1-e^{-q})^3}$$

$$\ell_{111} = \ell_{333} = \ell_{311} = \ell_{313} = \ell_{131} = \ell_{113} = \ell_{331} = \ell_{133} \quad (1.35)$$

Now, From equation number (1.31) to (1.35)

$$-[\ell_{ijk}] = \begin{bmatrix} \ell_{111} & \ell_{112} & \ell_{113} \\ \ell_{221} & \ell_{222} & \ell_{223} \\ \ell_{331} & \ell_{332} & \ell_{333} \end{bmatrix}$$

$$-\ell_{ijk} =$$

$$\rho_2 = \frac{\partial \rho}{\partial \beta} = \frac{\xi+1}{\beta} - \frac{1}{\gamma}$$

$$\begin{bmatrix} \beta^3 q - 3\beta^3 q^2 + 2\beta^3 q^3 + \frac{\beta^3 q e^{-q}(1-3q+2q^2)}{1-e^{-q}} - \frac{3\beta^3 q^2 e^{-2q}(1-q)}{(1-e^{-q})^2} + \frac{2\beta^3 q^3 e^{-3q}}{(1-e^{-q})^3} & \beta^2 u q - 2\beta^2 u q^2 + \frac{\beta^2 u q e^{-q}(1-2q)}{1-e^{-q}} - \frac{2\beta^2 u q^2 e^{-2q}}{(1-e^{-q})^2} & \beta^3 q - 3\beta^3 q^2 + 2\beta^3 q^3 + \frac{\beta^3 q e^{-q}(1-3q+2q^2)}{1-e^{-q}} - \frac{3\beta^3 q^2 e^{-2q}(1-q)}{(1-e^{-q})^2} + \frac{2\beta^3 q^3 e^{-3q}}{(1-e^{-q})^3} \\ \beta^2 u q - 2\beta^2 u q^2 + \frac{\beta^2 u q e^{-q}(1-2q)}{1-e^{-q}} - \frac{2\beta^2 u q^2 e^{-2q}}{(1-e^{-q})^2} & \frac{2}{\beta^3} + u^3 q - 3u^3 q^2 + 2u^3 q^3 + \frac{u^3 q e^{-q}(1-3q+2q^2)}{1-e^{-q}} - \frac{3u^3 q^2 e^{-2q}(1-q)}{(1-e^{-q})^2} + \frac{2u^3 q^3 e^{-3q}}{(1-e^{-q})^3} & \beta^2 u q - 2\beta^2 u q^2 + \frac{\beta^2 u q e^{-q}(1-2q)}{1-e^{-q}} - \frac{2\beta^2 u q^2 e^{-2q}}{(1-e^{-q})^2} \\ \beta^3 q - 3\beta^3 q^2 + 2\beta^3 q^3 + \frac{\beta^3 q e^{-q}(1-3q+2q^2)}{1-e^{-q}} - \frac{3\beta^3 q^2 e^{-2q}(1-q)}{(1-e^{-q})^2} + \frac{2\beta^3 q^3 e^{-3q}}{(1-e^{-q})^3} & \beta^2 u q - 2\beta^2 u q^2 + \frac{\beta^2 u q e^{-q}(1-2q)}{1-e^{-q}} - \frac{2\beta^2 u q^2 e^{-2q}}{(1-e^{-q})^2} & \beta^3 q - 3\beta^3 q^2 + 2\beta^3 q^3 + \frac{\beta^3 q e^{-q}(1-3q+2q^2)}{1-e^{-q}} - \frac{3\beta^3 q^2 e^{-2q}(1-q)}{(1-e^{-q})^2} + \frac{2\beta^3 q^3 e^{-3q}}{(1-e^{-q})^3} \end{bmatrix}$$

(1.36)

$$= \begin{bmatrix} K_{11} K_{12} K_{13} \\ K_{21} K_{22} K_{23} \\ K_{31} K_{32} K_{33} \end{bmatrix}$$

Determinantal value of $-\ell_{ijk}$

$$D = -\left\{ \begin{matrix} K_{11}(K_{22}K_{33} - K_{23}K_{32}) + K_{12}(K_{21}K_{33} - K_{23}K_{31}) + \\ (K_{21}K_{32} - K_{31}K_{22}) \end{matrix} \right\}$$

$$\text{Adjoint of matrix} = - \begin{bmatrix} \ell_{ijk} \end{bmatrix}$$

$$\text{Cofactor of matrix} = - \begin{bmatrix} \ell_{ijk} \end{bmatrix}$$

$$a_{11} = [K_{22}K_{33} - K_{23}K_{32}] = M_{11}$$

$$a_{12} = -[K_{21}K_{33} - K_{23}K_{31}] = K_{23}K_{31} - K_{21}K_{33} = M_{12}$$

$$a_{13} = [K_{21}K_{32} - K_{22}K_{31}] = M_{13}$$

$$a_{21} = [K_{12}K_{33} - K_{32}K_{13}] = K_{32}K_{13} - K_{12}K_{33} = M_{21}$$

$$a_{22} = [K_{12}K_{33} - K_{31}K_{13}] = M_{22}$$

$$a_{23} = [K_{11}K_{32} - K_{12}K_{31}] = K_{12}K_{31} - K_{11}K_{32} = M_{23}$$

$$a_{31} = [K_{12}K_{23} - K_{13}K_{22}] = M_{31}$$

$$a_{32} = -[K_{11}K_{23} - K_{13}K_{21}] = [K_{13}K_{21} - K_{11}K_{23}] = M_{32}$$

$$a_{33} = [K_{12}K_{22} - K_{12}K_{21}] = M_{33}$$

$$[-\ell_{ijk}]^{-1} = \frac{\text{Adjoint of } [\ell_{ijk}]}{|-\ell_{ijk}|} =$$

$$\begin{bmatrix} \frac{M_{11}}{D} & \frac{M_{21}}{D} & \frac{M_{13}}{D} \\ \frac{M_{12}}{D} & \frac{M_{22}}{D} & \frac{M_{23}}{D} \\ \frac{M_{13}}{D} & \frac{M_{23}}{D} & \frac{M_{33}}{D} \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11}\sigma_{21}\sigma_{13} \\ \sigma_{12}\sigma_{22}\sigma_{23} \\ \sigma_{13}\sigma_{23}\sigma_{33} \end{bmatrix}$$

Approximate Bayes Estimator Approach

$$V(\alpha, \beta, \gamma) = v$$

$$\hat{V}_{AB} = E(v|\underline{x})$$

evaluated from equation number (1.19) and (1.22)

joint prior density

$$J(\alpha, \beta, \gamma) = j_1(\alpha)j_2(\gamma)j_3\left(\frac{\beta}{\gamma}\right)$$

$$= \frac{c}{\partial \Gamma \xi} \gamma^{-\xi} \beta^{\xi+1} \exp \left[-\left(\frac{\gamma}{\delta} + \frac{\beta}{\gamma} \right) \right]$$

$$\rho = \log J = \log c - \log \partial - \log \Gamma \xi + (\xi +$$

$$1) \log \beta - \xi \log \gamma - \left(\frac{\gamma}{\delta} + \frac{\beta}{\gamma} \right)$$

$$\rho_1 = \frac{\partial \rho}{\partial \alpha} = 0$$

Then substituting all the above value in S, T and A we obtain,

$$S = [\sigma_{11}\ell_{111} + 2\sigma_{12}\ell_{121} + 2\sigma_{13}\ell_{131} + 2\sigma_{23}\ell_{231} + \sigma_{22}\ell_{221} + \sigma_{33}\ell_{331}]$$

$$\begin{aligned}
&= \sigma_{11} \left(\beta^3 q - 3\beta^3 q^2 + 2\beta^3 q^3 \right. \\
&\quad + \frac{\square^3 \square \square \square^{-\square} (I - 3\square + 2\square^2)}{3\square^3 \square^2 \square^{-2\square} (I - \square)} \\
&\quad - \frac{I - \square^{-\square}}{(I - \square^{-\square})^2} \\
&\quad + \left. \frac{2\beta^3 q^3 e^{-3q}}{(I - e^{-q})^3} \right) \\
&\quad + 2\sigma_{12} \left(\beta^2 u q - 2\beta^2 u q^2 \right. \\
&\quad + \frac{\square^2 \square \square \square \square^{-\square} (I - 2\square)}{I - \square^{-\square}} \\
&\quad - \left. \frac{2\beta^2 u q^2 e^{-2q}}{(I - e^{-q})^2} \right) \\
&\quad + 2\sigma_{13} \left(\beta^3 q - 3\beta^3 q^2 + 2\beta^3 q^3 \right. \\
&\quad + \frac{\square^3 \square \square \square^{-\square} (I - 3\square + 2\square^2)}{I - \square^{-\square}} \\
&\quad - \frac{3\square^3 \square^2 \square^{-2\square} (I - \square)}{(I - \square^{-\square})^2} \\
&\quad + \left. \frac{2\beta^3 q^3 e^{-3q}}{(I - e^{-q})^3} \right) \\
&\quad + 2\sigma_{23} \left(\beta^2 u q - 2\beta^2 u q^2 \right. \\
&\quad + \frac{\square^2 \square \square \square \square^{-\square} (I - 2\square)}{I - \square^{-\square}} \\
&\quad - \left. \frac{2\beta^2 u q^2 e^{-2q}}{(I - e^{-q})^2} \right) \\
&\quad + \sigma_{22} \left(\beta^2 u q - 2\beta^2 u q^2 \right. \\
&\quad + \frac{\square^2 \square \square \square \square^{-\square} (I - 2\square)}{I - \square^{-\square}} \\
&\quad - \left. \frac{2\beta^2 u q^2 e^{-2q}}{(I - e^{-q})^2} \right) \\
&\quad + \sigma_{33} \left(\beta^3 q - 3\beta^3 q^2 + 2\beta^3 q^3 \right. \\
&\quad + \frac{\square^3 \square \square \square^{-\square} (I - 3\square + 2\square^2)}{I - \square^{-\square}} \\
&\quad - \frac{3\square^3 \square^2 \square^{-2\square} (I - \square)}{(I - \square^{-\square})^2} \\
&\quad + \left. \frac{2\beta^3 q^3 e^{-3q}}{(I - e^{-q})^3} \right) \\
&\quad + 2\sigma_{12} \ell_{112} + 2\sigma_{13} \ell_{132} + 2\sigma_{23} \ell_{232} + \\
&\quad + \sigma_{22} \ell_{222} + \sigma_{33} \ell_{332}]
\end{aligned}$$

$$\begin{aligned}
&= \sigma_{11} \left(\beta^2 u q - 2\beta^2 u q^2 + \frac{\beta^2 u q e^{-q} (1-2q)}{1-e^{-q}} \right. \\
&\quad \left. - \frac{2\beta^2 u q^2 e^{-2q}}{(1-e^{-q})^2} \right) \\
&\quad + 2\sigma_{12} \left(\beta^2 u q - 2\beta^2 u q^2 \right. \\
&\quad \left. + \frac{\beta^2 u q e^{-q} (1-2q)}{1-e^{-q}} - \frac{2\beta^2 u q^2 e^{-2q}}{(1-e^{-q})^2} \right) \\
&\quad + 2\sigma_{13} \left(2\beta \frac{1-R}{R} \left(p - \frac{p^2}{R} \right) \right. \\
&\quad \left. + \beta^2 c \left[-p \frac{1-R}{R^2} \left(p - \frac{p^2}{R} \right) \right. \right. \\
&\quad \left. \left. + \frac{1-R}{R} \left(-p \right. \right. \right. \\
&\quad \left. \left. \left. + \frac{2p^2 R + p^3 (1-R)}{R^2} \right) \right] \right) \\
&\quad + 2\sigma_{23} \left(\beta^2 u q - 2\beta^2 u q^2 \right. \\
&\quad \left. + \frac{\beta^2 u q e^{-q} (1-2q)}{1-e^{-q}} - \frac{2\beta^2 u q^2 e^{-2q}}{(1-e^{-q})^2} \right) \\
&\quad + \sigma_{22} \left(\frac{2}{\beta^3} + u^3 q - 3u^3 q^2 + 2u^3 q^3 \right. \\
&\quad \left. + \frac{\square^3 \square \square \square^{-\square} (1-3\square + 2\square^2)}{1-\square^{-\square}} \right. \\
&\quad \left. - \frac{3\square^3 \square^2 \square^{-2\square} (1-\square)}{(1-\square^{-\square})^2} \right. \\
&\quad \left. + \frac{2u^3 q^3 e^{-3q}}{(1-e^{-q})^3} \right) \\
&\quad + \sigma_{33} \left(\beta^2 u q - 2\beta^2 u q^2 \right. \\
&\quad \left. + \frac{\beta^2 u q e^{-q} (1-2q)}{1-e^{-q}} - \frac{2\beta^2 u q^2 e^{-2q}}{(1-e^{-q})^2} \right) \\
&A = [\sigma_{11} \ell_{113} + 2\sigma_{12} \ell_{123} + 2\sigma_{13} \ell_{133} + 2\sigma_{23} \ell_{233} + \\
&\sigma_{22} \ell_{223} + \sigma_{33} \ell_{333}] \\
&= \sigma_{11} \left(\beta^3 q - 3\beta^3 q^2 + 2\beta^3 q^3 + \frac{\beta^3 q e^{-q} (1-3q+2q^2)}{1-e^{-q}} - \right. \\
&\quad \left. \frac{3\beta^3 q^2 e^{-2q} (1-q)}{(1-e^{-q})^2} + \frac{2\beta^3 q^3 e^{-3q}}{(1-e^{-q})^3} \right) + 2\sigma_{12} \left(\beta^2 u q - \right. \\
&\quad \left. 2\beta^2 u q^2 + \frac{\beta^2 u q e^{-q} (1-2q)}{1-e^{-q}} - \frac{2\beta^2 u q^2 e^{-2q}}{(1-e^{-q})^2} \right) + \\
&\quad 2\sigma_{13} \left(\beta^3 q - 3\beta^3 q^2 + 2\beta^3 q^3 + \frac{\beta^3 q e^{-q} (1-3q+2q^2)}{1-e^{-q}} - \right. \\
&\quad \left. \frac{3\beta^3 q^2 e^{-2q} (1-q)}{(1-e^{-q})^2} + \frac{2\beta^3 q^3 e^{-3q}}{(1-e^{-q})^3} \right) + 2\sigma_{23} \left(\beta^2 u q - \right. \\
&\quad \left. 2\beta^2 u q^2 + \frac{\beta^2 u q e^{-q} (1-2q)}{1-e^{-q}} - \frac{2\beta^2 u q^2 e^{-2q}}{(1-e^{-q})^2} \right) + \\
&\quad \sigma_{22} \left(\beta^2 u q - 2\beta^2 u q^2 + \frac{\beta^2 u q e^{-q} (1-2q)}{1-e^{-q}} - \right.
\end{aligned}$$

$$\frac{2\beta^2 u q^2 e^{-2q}}{(1-e^{-q})^2} + \sigma_{33} \left(\beta^3 q - 3\beta^3 q^2 + 2\beta^3 q^3 + \frac{\beta^3 q e^{-q}(1-3q+2q^2)}{1-e^{-q}} - \frac{3\beta^3 q^2 e^{-2q}(1-q)}{(1-e^{-q})^2} + \frac{2\beta^3 q^3 e^{-3q}}{(1-e^{-q})^3} \right)$$

$$\hat{V}_{AB} = E(v|\underline{x}) = v + (v_1\alpha_1 + v_2\alpha_2 + v_3\alpha_3 + \alpha_4 + \alpha_5) + \frac{1}{2}[(S\sigma_{11} + T\sigma_{21} + A\sigma_{31})v_1 + (S\sigma_{12} + T\sigma_{22} + A\sigma_{32})v_2 + (S\sigma_{13} + T\sigma_{23} + A\sigma_{33})v_3]$$

$$= v + \phi_1 + \phi_2$$

$$\phi_1 = (v_1\alpha_1 + v_2\alpha_2 + v_3\alpha_3 + \alpha_4 + \alpha_5)$$

$$\phi_2 = \frac{1}{2}[(S\sigma_{11} + T\sigma_{21} + A\sigma_{31})v_1 + (S\sigma_{12} + T\sigma_{22} + A\sigma_{32})v_2 + (S\sigma_{13} + T\sigma_{23} + A\sigma_{33})v_3]$$

evaluated at the MLE $\hat{V} = (\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ where;

$$\alpha_1 = \rho_1\sigma_{11} + \rho_2\sigma_{12} + \rho_3\sigma_{13}$$

$$= \sigma_{11} + \left[\frac{\xi+1}{\beta} - \frac{1}{\gamma}\right]\sigma_{12} + \left[\frac{-\xi}{\gamma} - \frac{1}{\delta} + \frac{1}{\gamma^2}\right]\sigma_{13}$$

$$\alpha_2 = \rho_1\sigma_{21} + \rho_2\sigma_{22} + \rho_3\sigma_{23}$$

$$= \left[\frac{\xi+1}{\beta} - \frac{1}{\gamma}\right]\sigma_{22} + \left[\frac{-\xi}{\gamma} - \frac{1}{\delta} + \frac{1}{\gamma^2}\right]\sigma_{23}$$

$$\alpha_3 = \rho_1\sigma_{31} + \rho_2\sigma_{32} + \rho_3\sigma_{33}$$

$$= \left[\frac{\xi+1}{\beta} - \frac{1}{\gamma}\right]\sigma_{32} + \left[\frac{-\xi}{\gamma} - \frac{1}{\delta} + \frac{1}{\gamma^2}\right]\sigma_{33}$$

$$\alpha_4 = v_{12}\sigma_{12} + v_{13}\sigma_{13} + v_{23}\sigma_{23}$$

$$\alpha_5 = \frac{1}{2}(v_{11}\sigma_{11} + v_{22}\sigma_{22} + v_{33}\sigma_{33})$$

Approximate Bayes Estimate of three parameter Gumbel distribution for Hazard rate function under LINEX loss function.

$$\hat{H}_{ABL} = -\frac{1}{k} \log [E_h(e^{-kH})]$$

$$\hat{H}_{ABL} = \frac{\iiint e^{-kH} J^*(\alpha, \beta, \gamma) d\alpha d\beta d\gamma}{\iiint J^*(\alpha, \beta, \gamma) d\alpha d\beta d\gamma}$$

The Lindley Approximation technique is applied to simplify the equation

$$\hat{H}_{ABL} = E[v|\underline{x}]$$

$$\approx e^{-kH} + \phi_5 + \phi_6$$

$$V = e^{-kH} \quad ; \quad H = \left[\frac{\beta e^{-\beta u} e^{-e^{-\beta u}}}{1 - e^{-e^{-\beta u}}} \right]$$

$$\log V = \log(e^{-kH})$$

$$\log V = -\frac{k\beta e^{-\beta u} e^{-e^{-\beta u}}}{1 - e^{-e^{-\beta u}}}$$

$$v_1 = 0$$

$$v_2 = -k e^{-kH} H_\beta \quad ; \quad H_\beta =$$

$$\frac{\partial}{\partial \beta} \left[\frac{\beta e^{-\beta u} e^{-e^{-\beta u}}}{1 - e^{-e^{-\beta u}}} \right]$$

$$v_3 = 0$$

$$v_{11} = 0$$

$$v_{12} = v_{21} = 0$$

$$v_{13} = v_{31} = 0$$

$$v_{22} = e^{-kH} [k^2 (H_\beta)^2 - k H_{\beta\beta}]$$

$$v_{23} = v_{32} = 0$$

$$v_{33} = 0$$

$$E[e^{-kH}|\underline{x}] \approx e^{-kH} + \phi_5 + \phi_6$$

$$\hat{H}_{ABL} = -\frac{1}{k} \log(e^{-kH} + \phi_5 + \phi_6)$$

$$\phi_5 = (v_1\alpha_1 + v_2\alpha_2 + v_3\alpha_3 + \alpha_4 + \alpha_5)$$

$$\phi_6 = \frac{1}{2}[(S\sigma_{11} + T\sigma_{21} + A\sigma_{31})v_1 + (S\sigma_{12} + T\sigma_{22} + A\sigma_{32})v_2 + (S\sigma_{13} + T\sigma_{23} + A\sigma_{33})v_3]$$

$$\phi_5 = 0\alpha_1 + (-k e^{-kH} H_\beta)\alpha_2 - 0\alpha_3 + 0\sigma_{12} +$$

$$0\sigma_{13} + 0\sigma_{23} + \frac{1}{2} \left(0\sigma_{11} + e^{-kH} [k^2 (H_\beta)^2 - k H_{\beta\beta}] \sigma_{22} + 0\sigma_{33} \right)$$

$$\phi_5 = e^{-kH} [-k H_\beta \alpha_2 + \frac{1}{2} (k^2 H_\beta^2 - k H_{\beta\beta}) \sigma_{22}]$$

$$\forall \Delta'_I = \{kp(1-R)(-\beta\alpha_1 + c\alpha_2 - \beta\alpha_3) + (k^2 p^2 (1-R)^2 - kp(1-R)(1-p))[-\beta c(\sigma_{12} + \sigma_{23}) + \beta^2 \sigma_{13} + \frac{1}{2}(\beta^2(\sigma_{11} + \sigma_{33}) + c^2 \sigma_{22})]\}$$

$$\phi_5 = e^{-kH} \Delta'_I$$

$$\phi_6 = \frac{1}{2}[(S\sigma_{11} + T\sigma_{21} + A\sigma_{31})0 + (S\sigma_{12} + T\sigma_{22} + A\sigma_{32}) - k e^{-kH} H_\beta + (S\sigma_{13} + T\sigma_{23} + A\sigma_{33})0]$$

$$\phi_6 = -\frac{k}{2} e^{-kH} H_\beta (S\sigma_{12} + T\sigma_{22} + A\sigma_{32})$$

$$\Delta'_2 = (S\sigma_{12} + T\sigma_{22} + A\sigma_{32})$$

$$\phi_6 = -\frac{k}{2} e^{-kH} H_\beta \Delta'_2$$

$$\hat{H}_{ABL} =$$

$$-\frac{1}{k} \log \left(e^{-kH} + e^{-kH} \Delta'_I + \left(-\frac{k}{2} e^{-kH} H_\beta \Delta'_2 \right) \right)$$

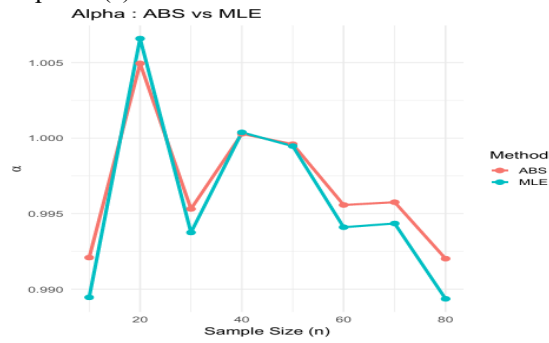
$$\hat{H}_{ABL} = H - \frac{1}{k} \log \left(1 + \Delta'_I - \frac{k}{2} H_\beta \Delta'_2 \right)$$

This is the estimation of Hazard rate function under LLF.

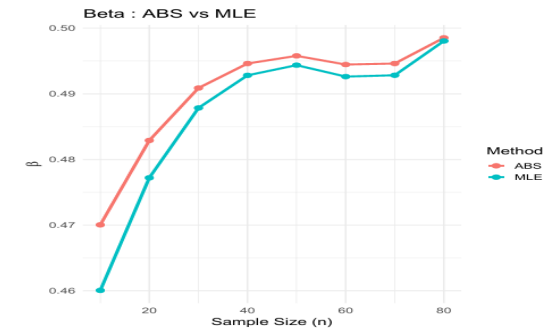
Table 1.1: MSEs and mean estimates for α , β , and γ under ABS and MLE

Bayesian Estimation of Hazard and Reverse Hazard Rates for the three parameters Gumbel Distribution under LINEX loss (LLF) function

Graph 1.1(a)

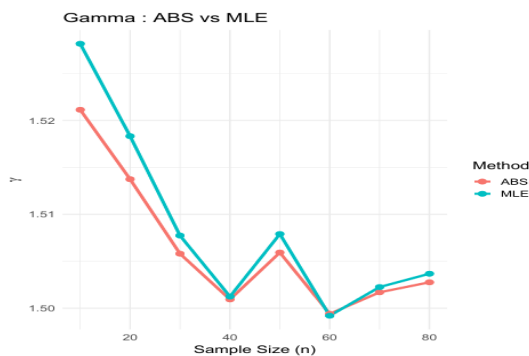


Graph 1.1(b)



Graph 1.1(c)

n	$\hat{\alpha}_{ABS}$	$\hat{\alpha}_{MLE}$	$\hat{\beta}_{ABS}$	$\hat{\beta}_{MLE}$	$\hat{\gamma}_{ABS}$	$\hat{\gamma}_{MLE}$
120	0.99209 56 [0.0078 51990]	0.9894 56 [0.013 95909]	0.47004 5 [0.0087 573691]	0.46006 0 [0.0155 68656]	1.52113 7 [0.0173 37269]	1.52818 3 [0.0308 21811]
128	1.00494 8 [0.0082 56824]	1.0065 98 [0.014 67880]	0.48290 3 [0.0045 668378]	0.47720 5 [0.0081 18823]	1.51374 2 [0.0072 21386]	1.51832 2 [0.0128 38019]
136	0.99531 6 [0.0076 06899]	0.9937 54 [0.013 52338]	0.49088 0 [0.0026 206548]	0.48784 0 [0.0046 58942]	1.50579 8 [0.0053 175591]	1.50773 1 [0.0094 53494]
147	1.00028 7 [0.0080 48981]	1.0003 83 [0.014 30930]	0.49461 2 [0.0020 094450]	0.49281 6 [0.0035 72347]	1.50091 8 [0.0042 44088]	1.50122 4 [0.0075 45045]
156	0.99960 6 [0.0073 75063]	0.9994 74 [0.013 11122]	0.49577 5 [0.0020 538935]	0.49436 6 [0.0036 51366]	1.50592 4 [0.0031 10101]	1.50789 9 [0.0055 29068]
168	0.99557 8 [0.0081 63081]	0.9941 05 [0.014 51214]	0.49446 3 [0.0015 065074]	0.49261 7 [0.0026 78235]	1.49938 8 [0.0024 89380]	1.49918 4 [0.0044 25564]
172	0.99576 2 [0.0076 05821]	0.9943 50 [0.013 52146]	0.49461 6 [0.0012 420620]	0.49282 1 [0.0022 08110]	1.50168 1 [0.0023 92051]	1.50224 2 [0.0042 52535]
182	0.99202 2 [0.0089 00105]	0.9893 62 [0.015 82241]	0.49853 4 [0.0009 871706]	0.49804 5 [0.0017 54970]	1.50274 5 [0.0019 80408]	1.50366 0 [0.0035 20724]



According to Table 1.1 and Graph 1.1(a)–1.1(c), the Approximate Bayes Estimators under the LINEX Loss Function (LLF) are compared with the Maximum Likelihood Estimator (MLE) for the parameters α , β , and γ . The numerical results presented in Table 1.1 indicate that the Mean Squared Error (MSE) of the Approximate Bayes Estimators is lower than that of the Maximum Likelihood Estimator. The graphical results also show that the Approximate Bayes Estimators remain closer to the true parameter values than the Maximum Likelihood Estimator. Hence, it can be concluded that the Approximate Bayes Estimation under LLF provides more accurate parameter estimates than the Maximum Likelihood Estimation method.

2.1 The reverse hazard rate function corresponding to the three-parameter Gumbel distribution is given by

$$H^*(t) = \frac{f(t)}{F(t)} = \frac{\beta e^{-(\beta(t-\gamma-\alpha))} e^{-e^{-(\beta(t-\gamma-\alpha))}}}{e^{-e^{-(\beta(t-\gamma-\alpha))}}}$$

(2.1)

$$\forall \text{Let, } u = t - \gamma - \alpha$$

$$H^*(t) = \frac{\beta e^{-\beta u} e^{-e^{-\beta u}}}{e^{-e^{-\beta u}}}$$

$$H^*(t) = \beta e^{-\beta u}$$

(2.2)

Then taking logarithms on both side of the likelihood function, we obtain

$$\log H^*(t) = \log[\beta e^{-\beta u}] \quad y =$$

$$\log H^*(t) = \log[\beta y]$$

$$\log H^*(t) = \log \beta + \log y$$

(2.3)

$$H^*(t) = \beta e^{-\beta u}$$

By this we find the value of β

$$\beta = \frac{H^*(t)}{e^{-\beta u}}$$

$$\beta = \frac{H^*(t)}{y}$$

$$\text{Derivative } \frac{d\beta}{dH^*(t)}$$

$$\text{Differentiating w.r.t } H^*(t)$$

$$\frac{d\beta}{dH^*(t)} = \frac{1}{y}$$

The posterior distribution obtained above is for the three-parameter Gumbel distribution based on the sample observations $(x_1, x_2, \dots, x_n)$. Since this

chapter deals with the reverse hazard rate function, we use (t) as the time variable instead of (x) . Therefore, the same posterior distribution is considered by replacing (x) with (t) , and the corresponding values obtained above are substituted to derive the Bayes estimators.

$$h(\beta | \underline{x}) =$$

$$\frac{\prod_{i=1}^n [\beta e^{-(\beta(x-\gamma-\alpha))} e^{-e^{-(\beta(x-\gamma-\alpha))}}] \frac{b^a}{\Gamma(a)} \beta^{a-1} e^{-\beta b}}{\int_0^\infty \prod_{i=1}^n [\beta e^{-(\beta(x-\gamma-\alpha))} e^{-e^{-(\beta(x-\gamma-\alpha))}}] \frac{b^a}{\Gamma(a)} \beta^{a-1} e^{-\beta b} d\beta}$$

Then, the above posterior distribution is converted into the following form

$$h(H^* | t) = \frac{\prod_{i=1}^n \left[\frac{H^*(t)}{e^{-\beta u}} e^{-\frac{H^*(t)}{e^{-\beta u}}(t-\gamma-\alpha)} e^{-e^{-\frac{H^*(t)}{e^{-\beta u}}(t-\gamma-\alpha)}} \right] \frac{b^a}{\Gamma(a)} \left(\frac{H^*(t)}{e^{-\beta u}} \right)^{a-1} e^{-\frac{H^*(t)}{e^{-\beta u}} b}}{\int_0^\infty \prod_{i=1}^n \left[\frac{H^*(t)}{e^{-\beta u}} e^{-\frac{H^*(t)}{e^{-\beta u}}(t-\gamma-\alpha)} e^{-e^{-\frac{H^*(t)}{e^{-\beta u}}(t-\gamma-\alpha)}} \right] \frac{b^a}{\Gamma(a)} \left(\frac{H^*(t)}{e^{-\beta u}} \right)^{a-1} e^{-\frac{H^*(t)}{e^{-\beta u}} b} dH^*(t)}$$

$$h(H^* | t) = \frac{\left(\frac{H^*(t)}{e^{-\beta u}} \right)^{n+a-1} e^{-\frac{H^*(t)}{e^{-\beta u}}(u+b)} e^{-ne^{-\frac{H^*(t)}{e^{-\beta u}}(u)}} \frac{b^a}{\Gamma(a)}}{\int_0^\infty \left(\frac{H^*(t)}{e^{-\beta u}} \right)^{n+a-1} e^{-\frac{H^*(t)}{e^{-\beta u}}(u+b)} e^{-ne^{-\frac{H^*(t)}{e^{-\beta u}}(u)}} \frac{b^a}{\Gamma(a)} dH^*(t)}$$

Now under asymptotic approximation:

$$e^{-e^{-z}} \approx 1$$

$$\therefore, e^{-ne^{-\frac{H^*(t)}{e^{-\beta u}}(u)}} \approx 1$$

$$h(H^* | t) = \frac{\left(\frac{H^*(t)}{e^{-\beta u}} \right)^{n+a-1} e^{-\frac{H^*(t)}{e^{-\beta u}}(u+b)}}{\int_0^\infty \left(\frac{H^*(t)}{e^{-\beta u}} \right)^{n+a-1} e^{-\frac{H^*(t)}{e^{-\beta u}}(u+b)} dH^*(t)}$$

after solving all this complex term we get,

$$h(H^*(t) | t) = \frac{\left(\frac{u+b}{e^{-\beta u}} \right)^{n+a} (H^*(t))^{n+a-1} e^{-\frac{u+b}{e^{-\beta u}} H^*(t)}}{\Gamma(n+a)}$$

$$\lambda = \left(\frac{u+b}{e^{-\beta u}} \right)$$

$$h(H^*(t) | t) = \frac{\lambda^{n+a} (H^*(t))^{n+a-1} e^{-\lambda H^*(t)}}{\Gamma(n+a)}$$

(2.4)

Bayes estimator under LINEX loss function for the Reverse Hazard rate function.

For the Bayes estimator under the Linear Exponential Loss Function (LLF), the posterior distribution is substituted into $\left[-\frac{1}{k} \log E(e^{-kH^*(t)} | t) \right]$, and the resulting expression is simplified as follows.

$$\hat{H}_{BL}^* = \left[-\frac{1}{k} \log E(e^{-kH^*(t)} | t) \right]$$

(2.5)

$$\begin{aligned} E(e^{-kH^*(t)}) &= \int_0^\infty e^{-kH^*(t)} h(H^*(t) | t) dH^*(t) \\ &= \int_0^\infty e^{-kH^*(t)} \frac{\lambda^{n+a} (H^*(t))^{n+a-1} e^{-\lambda H^*(t)}}{\Gamma(n+a)} dH^*(t) \\ &= \frac{\lambda^{n+a}}{\Gamma(n+a)} \int_0^\infty (H^*(t))^{n+a-1} e^{-(k+\lambda)H^*(t)} dH^*(t) \end{aligned}$$

Solve all this

$$\begin{aligned} \hat{H}_{BL}^* &= -\frac{1}{k} \log \left[\left(\frac{\lambda}{k+\lambda} \right)^{n+a} \right] \\ &= \frac{n+a}{k} \log \left(\frac{k+\lambda}{\lambda} \right) \end{aligned}$$

$$\hat{H}_{BL}^* = \frac{n+a}{k} \log \left[1 + \frac{ke^{-\beta u}}{u+b} \right] \quad (2.6)$$

The Bayesian approach to estimating the unknown parameters α, β and γ

In Bayesian analysis, prior distribution is used to incorporate prior knowledge about the parameter of a distribution. For the parameters α, β and γ of the three-parameter Gumbel distribution, the following prior structure is assumed.

Then, combined prior distribution corresponding to the parameters α, β , and γ can be expressed as

$$J(\alpha, \beta, \gamma) = j_1(\alpha)j_2(\gamma)j_3(\beta|\gamma)$$

Taking

$$j_1(\alpha) = c$$

$$(2.7)$$

$$j_2(\gamma) = \frac{1}{\delta} e^{-\frac{\gamma}{\delta}}$$

$$(2.8)$$

$$j_3(\beta|\gamma) = \frac{1}{\Gamma\xi} \gamma^{-\xi} \beta^{\xi+1} e^{-\frac{\beta}{\gamma}}$$

$$(2.9)$$

$$J(\alpha, \beta, \gamma) = j_1(\alpha)j_2(\gamma)j_3(\beta|\gamma) \\ = \frac{c}{\delta\Gamma\xi} \gamma^{-\xi} \beta^{\xi+1} e^{-\left(\frac{\gamma}{\delta} + \frac{\beta}{\gamma}\right)}$$

$$(2.10)$$

Joint posterior with likelihood equation number (2.3) and (2.10) we get;

$$h^*(\alpha, \beta, \gamma) = \frac{\gamma^{-\xi} \beta^{\xi+1} e^{-\left(\frac{\gamma}{\delta} + \frac{\beta}{\gamma}\right)} L(\underline{x}|\alpha, \beta, \gamma)}{\iiint \gamma^{-\xi} \beta^{\xi+1} e^{-\left(\frac{\gamma}{\delta} + \frac{\beta}{\gamma}\right)} L(\underline{x}|\alpha, \beta, \gamma) d\alpha d\beta d\gamma}$$

$$(2.11)$$

The approximate Bayes estimators

$$V(\Theta) = v(\alpha, \beta, \gamma)$$

$$\hat{v}_{AB} = E(v|\underline{x}) = \frac{\iiint v(\alpha, \beta, \gamma) J^*(\alpha, \beta, \gamma) d\alpha d\beta d\gamma}{\iiint J^*(\alpha, \beta, \gamma) d\alpha d\beta d\gamma}$$

• Lindley Approximation

In statistical inference, while estimating the parameters of a distribution, we often obtain very complicated integral expressions in Bayesian estimation. These integrals are difficult to evaluate analytically. Therefore, Lindley's approximation is applied to simplify the estimation procedure. With the help of Lindley's approximation, Approximate Bayes Estimators can be obtained easily for the parameters or functions of a distribution.

$$E(v(\alpha, \beta, \gamma|\underline{x})) = V(\Theta) + \frac{1}{2} [S(v_{11}\sigma_{11} + v_{12}\sigma_{12} + v_{13}\sigma_{13}) + \\ T(v_{21}\sigma_{21} + v_{22}\sigma_{22} + v_{23}\sigma_{23}) + A(v_{31}\sigma_{31} + v_{32}\sigma_{32} + \\ v_{33}\sigma_{33}) + v_1 a_1 + v_2 a_2 + v_3 a_3 + a_4 + a_5 + O\left(\frac{1}{n^2}\right)] \quad (2.12)$$

Evaluated at MLE = $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ where;

$$a_1 = \rho_1 \sigma_{11} + \rho_2 \sigma_{12} + \rho_3 \sigma_{13}$$

$$(2.13)$$

$$a_2 = \rho_1 \sigma_{21} + \rho_2 \sigma_{22} + \rho_3 \sigma_{23}$$

$$(2.14)$$

$$a_3 = \rho_1 \sigma_{31} + \rho_2 \sigma_{32} + \rho_3 \sigma_{33}$$

$$(2.15)$$

$$a_4 = v_{12}\sigma_{12} + v_{13}\sigma_{13} + v_{23}\sigma_{23}$$

$$(2.16)$$

$$a_5 = \frac{1}{2} (v_{11}\sigma_{11} + v_{22}\sigma_{22} + v_{33}\sigma_{33}) \quad (2.17)$$

$$S = [\sigma_{11}\ell_{111} + 2\sigma_{12}\ell_{121} + 2\sigma_{13}\ell_{131} + 2\sigma_{23}\ell_{231} + \sigma_{22}\ell_{221} + \sigma_{33}\ell_{331}] \quad (2.18) \quad T =$$

$$[\sigma_{11}\ell_{112} + 2\sigma_{12}\ell_{122} + 2\sigma_{13}\ell_{132} + 2\sigma_{23}\ell_{232} + \sigma_{22}\ell_{222} + \sigma_{33}\ell_{332}] \quad (2.19) \quad A = [\sigma_{11}\ell_{113} +$$

$$2\sigma_{12}\ell_{123} + 2\sigma_{13}\ell_{133} + 2\sigma_{23}\ell_{233} + \sigma_{22}\ell_{223} + \sigma_{33}\ell_{333}] \quad (2.20)$$

For applying Lindley approximation to equation number (2.12)

The Likelihood function described by equation number (2.3)

$$\log H^*(t) = \log \beta + \log y$$

$$\ell = \log \beta + \log y$$

$$(2.21)$$

$$\ell = \log \beta - \beta u$$

$$(2.22)$$

$$\ell_1 = \frac{\partial \ell}{\partial \alpha} = \beta$$

$$\ell_2 = \frac{1}{\beta} - u$$

$$\ell_3 = \beta$$

$$\ell_{11} = 0$$

$$\ell_{12} = \ell_{21} = 1$$

$$\ell_{13} = \ell_{31} = 0$$

$$\ell_{22} = -\frac{1}{\beta^2}$$

$$\ell_{23} = \ell_{32} = 1$$

$$\ell_{33} = 0$$

$$\ell_{111} = \ell_{333} = 0$$

$$(2.23)$$

$$\ell_{222} = \frac{2}{\beta^3}$$

$$(2.24)$$

$$\ell_{121} = \ell_{211} = \ell_{112} = \ell_{332} = \ell_{323} = \ell_{233} = \ell_{223} = \ell_{123} = \ell_{132} = \ell_{213} = \ell_{231} = \ell_{321} = 0 \quad (2.25)$$

$$\ell_{111} = \ell_{333} = \ell_{311} = \ell_{313} = \ell_{131} = \ell_{113} = \ell_{331} = \ell_{133} = 0$$

$$(2.26)$$

Now, From equation number (2.23) to (2.26)

$$-[\ell_{ijk}] = \begin{bmatrix} \ell_{111} & \ell_{112} & \ell_{113} \\ \ell_{221} & \ell_{222} & \ell_{223} \\ \ell_{331} & \ell_{332} & \ell_{333} \end{bmatrix}$$

$$-[\ell_{ijk}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{2}{\beta^3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}$$

Determinantal value of $-[\ell_{ijk}]$

$$D = -\left\{ K_{11}(K_{22}K_{33} - K_{23}K_{32}) + K_{12}(K_{21}K_{33} - K_{23}K_{31}) + \right. \\ \left. (K_{21}K_{32} - K_{31}K_{22}) \right\}$$

$$\text{Adjoint of matrix} = -[\ell_{ijk}]$$

$$\text{Cofactor of matrix} = -[\ell_{ijk}]$$

$$a_{11} = [K_{22}K_{33} - K_{23}K_{32}] = M_{11}$$

$$a_{12} = -[K_{21}K_{33} - K_{23}K_{31}] \\ = K_{23}K_{31} - K_{21}K_{33} = M_{12}$$

$$a_{13} = [K_{21}K_{32} - K_{22}K_{31}] = M_{13}$$

$$\begin{aligned} a_{21} &= [K_{12}K_{33} - K_{32}K_{13}] \\ &= K_{32}K_{13} - K_{12}K_{33} = M_{21} \\ a_{22} &= [K_{12}K_{33} - K_{31}K_{13}] = M_{22} \\ a_{23} &= [K_{11}K_{32} - K_{12}K_{31}] \\ &= K_{12}K_{31} - K_{11}K_{32} = M_{23} \\ a_{31} &= [K_{12}K_{23} - K_{13}K_{22}] = M_{31} \\ a_{32} &= -[K_{11}K_{23} - K_{13}K_{21}] \\ &= [K_{13}K_{21} - K_{11}K_{23}] = M_{32} \\ a_{33} &= [K_{12}K_{22} - K_{12}K_{21}] = M_{33} \\ [-\ell_{ijk}]^{-1} &= \frac{\text{Adjoint of } [\ell_{ijk}]}{|-\ell_{ijk}|} = \end{aligned}$$

$$\begin{bmatrix} \frac{M_{11}}{D} & \frac{M_{21}}{D} & \frac{M_{13}}{D} \\ \frac{M_{12}}{D} & \frac{M_{22}}{D} & \frac{M_{23}}{D} \\ \frac{M_{13}}{D} & \frac{M_{23}}{D} & \frac{M_{33}}{D} \end{bmatrix} = \begin{bmatrix} \sigma_{11}\sigma_{21}\sigma_{13} \\ \sigma_{12}\sigma_{22}\sigma_{23} \\ \sigma_{13}\sigma_{23}\sigma_{33} \end{bmatrix}$$

Approximate Bayes Estimator Approach

$$V(\alpha, \beta, \gamma) = v$$

$$\hat{V}_{AB} = E(v|\underline{x})$$

evaluated from equation number (2.10) and (2.12) joint prior density

$$\begin{aligned} J(\alpha, \beta, \gamma) &= j_1(\alpha)j_2(\gamma)j_3\left(\frac{\beta}{\gamma}\right) \\ &= \frac{c}{\partial \Gamma \xi} \gamma^{-\xi} \beta^{\xi+1} \exp\left[-\left(\frac{\gamma}{\partial} + \frac{\beta}{\gamma}\right)\right] \\ \rho &= \log J = \log c - \log \partial - \log \Gamma \xi + (\xi + 1) \log \beta - \xi \log \gamma - \left(\frac{\gamma}{\partial} + \frac{\beta}{\gamma}\right) \end{aligned} \quad (2.27)$$

$$\rho_1 = \frac{\partial \rho}{\partial \alpha} = 0 \quad (2.28)$$

$$\rho_2 = \frac{\partial \rho}{\partial \beta} = \frac{\xi+1}{\beta} - \frac{1}{\gamma}$$

$$\rho_3 = \frac{\partial \rho}{\partial \gamma} = \frac{-\xi}{\gamma} - \frac{1}{\delta} + \frac{1}{\gamma^2} \quad (2.30)$$

Then substituting all the above value in S, T and A we obtain,

$$S = [\sigma_{11}\ell_{111} + 2\sigma_{12}\ell_{121} + 2\sigma_{13}\ell_{131} + 2\sigma_{23}\ell_{231} + \sigma_{22}\ell_{221} + \sigma_{33}\ell_{331}] \quad (2.31)$$

$$\sigma_{11}0 + 2\sigma_{12}0 + 2\sigma_{13}0 + 2\sigma_{23}0 + \sigma_{22}0 + \sigma_{33}0$$

$$\begin{aligned} T &= [\sigma_{11}\ell_{112} + 2\sigma_{12}\ell_{122} + 2\sigma_{13}\ell_{132} + 2\sigma_{23}\ell_{232} + \sigma_{22}\ell_{222} + \sigma_{33}\ell_{332}] \\ &\quad \sigma_{11}0 + 2\sigma_{12}0 + 2\sigma_{13}0 + 2\sigma_{23}0 + \sigma_{22}\frac{2}{\beta^3} + \sigma_{33}0 \end{aligned} \quad (2.32)$$

$$A = [\sigma_{11}\ell_{113} + 2\sigma_{12}\ell_{123} + 2\sigma_{13}\ell_{133} + 2\sigma_{23}\ell_{233} + \sigma_{22}\ell_{223} + \sigma_{33}\ell_{333}] \quad (2.33)$$

$$\sigma_{11}0 + 2\sigma_{12}0 + 2\sigma_{13}0 + 2\sigma_{23}0 + \sigma_{22}0 + \sigma_{33}0$$

$$\begin{aligned} \hat{V}_{AB} &= E(v|\underline{x}) = v + (v_1a_1 + v_2a_2 + v_3a_3 + \alpha_4 + \alpha_5) + \frac{1}{2}[(S\sigma_{11} + T\sigma_{21} + A\sigma_{31})v_1 + (S\sigma_{12} + T\sigma_{22} + A\sigma_{32})v_2 + (S\sigma_{13} + T\sigma_{23} + A\sigma_{33})v_3] \\ &\quad (2.34) \end{aligned}$$

$$= v + \phi_1 + \phi_2$$

$$(2.35)$$

$$\phi_1 = (v_1a_1 + v_2a_2 + v_3a_3 + \alpha_4 + \alpha_5)$$

$$(2.36)$$

$$\phi_2 = \frac{1}{2}[(S\sigma_{11} + T\sigma_{21} + A\sigma_{31})v_1 + (S\sigma_{12} + T\sigma_{22} + A\sigma_{32})v_2 + (S\sigma_{13} + T\sigma_{23} + A\sigma_{33})v_3] \quad (2.37)$$

evaluated at the MLE $\hat{V} = (\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ where;

$$\begin{aligned} a_1 &= \rho_1\sigma_{11} + \rho_2\sigma_{12} + \rho_3\sigma_{13} \\ &= 0.\sigma_{11} + \left[\frac{\xi+1}{\beta} - \frac{1}{\gamma}\right]\sigma_{12} + \left[\frac{-\xi}{\gamma} - \frac{1}{\delta} + \frac{1}{\gamma^2}\right]\sigma_{13} \end{aligned} \quad (2.38)$$

$$\begin{aligned} a_2 &= \rho_1\sigma_{21} + \rho_2\sigma_{22} + \rho_3\sigma_{23} \\ &= \left[\frac{\xi+1}{\beta} - \frac{1}{\gamma}\right]\sigma_{22} + \left[\frac{-\xi}{\gamma} - \frac{1}{\delta} + \frac{1}{\gamma^2}\right]\sigma_{23} \end{aligned} \quad (2.39)$$

$$\begin{aligned} a_3 &= \rho_1\sigma_{31} + \rho_2\sigma_{32} + \rho_3\sigma_{33} \\ &= \left[\frac{\xi+1}{\beta} - \frac{1}{\gamma}\right]\sigma_{32} + \left[\frac{-\xi}{\gamma} - \frac{1}{\delta} + \frac{1}{\gamma^2}\right]\sigma_{33} \end{aligned}$$

$$(2.40)$$

$$a_4 = v_{12}\sigma_{12} + v_{13}\sigma_{13} + v_{23}\sigma_{23} \quad (2.41)$$

$$a_5 = \frac{1}{2}(v_{11}\sigma_{11} + v_{22}\sigma_{22} + v_{33}\sigma_{33})$$

$$(2.42)$$

Approximate Bayes Estimate of three parameter Gumbel distribution for Reverse Hazard rate function under LINEX loss function.

$$\hat{H}_{ABL}^* = -\frac{1}{k} \log [E_h(e^{-kH^*})]$$

$$\hat{H}_{ABL}^* = \frac{\iiint e^{-kH^*} J^*(\alpha, \beta, \gamma) d\alpha d\beta d\gamma}{\iiint J^*(\alpha, \beta, \gamma) d\alpha d\beta d\gamma}$$

The Lindley Approximation technique is applied to simplify the equation

$$\hat{H}_{ABL}^* = E[v|\underline{x}] \approx e^{-kH^*} + \phi_5 + \phi_6$$

$$V = e^{-kH^*} ; \forall H^* = [\beta e^{-\beta u}]$$

$$\text{Log} V = \log(e^{-kH^*})$$

$$\log V = -k\beta e^{-\beta u}$$

$$v_1 = 0$$

$$v_2 = -ke^{-\beta u}(1 - \beta u)e^{-k\beta e^{-\beta u}}$$

$$v_3 = 0$$

$$v_{11} = 0$$

$$v_{12} = v_{21} = 0$$

$$v_{13} = v_{31} = 0$$

$$v_{22} = e^{-k\beta e^{-\beta u}}[-ke^{-\beta u}(\beta u^2 - 2u) + k^2e^{-2\beta u}(1 - \beta u)^2]$$

$$v_{23} = v_{32} = 0$$

$$v_{33} = 0$$

$$E[e^{-kH^*}|\underline{x}] \approx e^{-kH^*} + \phi_5 + \phi_6$$

$$\hat{H}_{ABL}^* = -\frac{1}{k} \log(e^{-kH^*} + \phi_5 + \phi_6)$$

$$\phi_5 = (v_1a_1 + v_2a_2 + v_3a_3 + \alpha_4 + \alpha_5)$$

$$\phi_6 = \frac{1}{2}[(S\sigma_{11} + T\sigma_{21} + A\sigma_{31})v_1 + (S\sigma_{12} + T\sigma_{22} + A\sigma_{32})v_2 + (S\sigma_{13} + T\sigma_{23} + A\sigma_{33})v_3]$$

$$\phi_5 = 0a_1 + (-ke^{-\beta u}(1 - \beta u)e^{-k\beta e^{-\beta u}})a_2 - 0a_3 +$$

$$\begin{aligned} &0\sigma_{12} + 0\sigma_{13} + 0\sigma_{23} + \frac{1}{2}\left(0\sigma_{11} + e^{-k\beta e^{-\beta u}}[-ke^{-\beta u}(\beta u^2 - 2u) + k^2e^{-2\beta u}(1 - \beta u)^2]\sigma_{22} + 0\sigma_{33}\right) \end{aligned}$$

$$\phi_5 = e^{-k\beta e^{-\beta u}}[-ke^{-\beta u}(1 - \beta u)a_2 +$$

$$\frac{1}{2}(-ke^{-\beta u}(\beta u^2 - 2u) + k^2e^{-2\beta u}(1 - \beta u)^2)\sigma_{22}]$$

$$\forall \Delta'_I = \left[-ke^{-\beta u}(1 - \beta u)a_2 + \frac{1}{2}(-ke^{-\beta u}(\beta u^2 - 2u) + k^2e^{-2\beta u}(1 - \beta u)^2)\sigma_{22} \right]$$

$$\varphi_5 = e^{-k\beta e^{-\beta u}} \Delta'_I$$

$$\varphi_6 = \frac{1}{2} \left[(S\sigma_{11} + T\sigma_{21} + A\sigma_{31})0 + (S\sigma_{12} + T\sigma_{22} + A\sigma_{32}) - ke^{-\beta u}(1 - \beta u)e^{-k\beta e^{-\beta u}} + (S\sigma_{13} + T\sigma_{23} + A\sigma_{33})0 \right]$$

$$\varphi_6 = -\frac{k}{2}e^{-\beta u}(1 - \beta u)e^{-k\beta e^{-\beta u}}(S\sigma_{12} + T\sigma_{22} + A\sigma_{32})$$

$$\Delta'_2 = (S\sigma_{12} + T\sigma_{22} + A\sigma_{32})$$

$$\varphi_6 = -\frac{k}{2}e^{-\beta u}(1 - \beta u)e^{-k\beta e^{-\beta u}} \Delta'_2$$

$$\varphi_5 + \varphi_6 = e^{-k\beta e^{-\beta u}} \Delta'_I + \left(-\frac{k}{2}e^{-\beta u}(1 - \beta u)e^{-k\beta e^{-\beta u}} \Delta'_2 \right)$$

$$\varphi_5 + \varphi_6 = e^{-k\beta e^{-\beta u}} \left[\Delta'_I - \frac{k}{2}e^{-\beta u}(1 - \beta u) \Delta'_2 \right]$$

$$\hat{H}_{ABL}^* = e^{-kH^*} + e^{-k\beta e^{-\beta u}} \left[\Delta'_I - \frac{k}{2}e^{-\beta u}(1 - \beta u) \Delta'_2 \right]$$

$$\hat{H}_{ABL}^* = e^{-k\beta e^{-\beta u}} \left[1 + \Delta'_I - \frac{k}{2}e^{-\beta u}(1 - \beta u) \Delta'_2 \right]$$

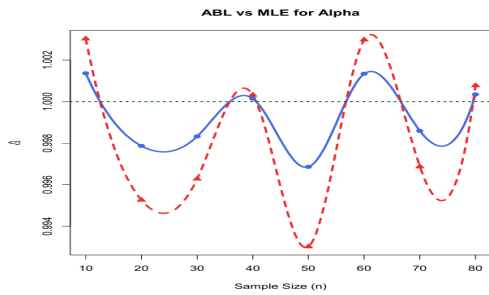
$$\hat{H}_{ABL}^* = \beta e^{-\beta u} - \frac{1}{k} \log \left(1 + \Delta'_I - \frac{k}{2}e^{-\beta u}(1 - \beta u) \Delta'_2 \right)$$

Table 2.1: MSEs and Mean Estimates for α , β , and γ under ABE and MLE

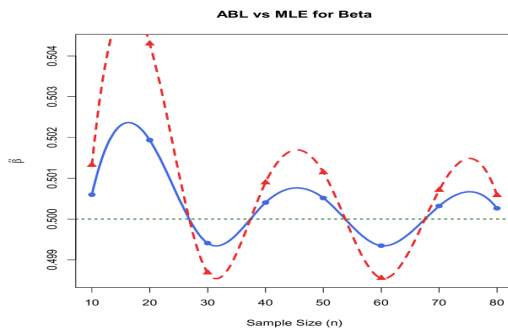
Graph 2.1 (a)

n	$\hat{\alpha}_{ABL}$	$\hat{\alpha}_{MLE}$	$\hat{\beta}_{ABL}$	$\hat{\beta}_{MLE}$	$\hat{\gamma}_{ABL}$	$\hat{\gamma}_{MLE}$
10	1.0013 62 [0.0037 02]	1.0030 26 [0.0182 80]	0.5005 97 [0.0020 47]	0.5013 26 [0.0101 11]	1.5018 81 [0.0064 05]	1.50376 2 [0.0256 22]
20	0.9978 68 [0.0020 33]	0.9952 62 [0.0100 41]	0.5019 36 [0.0009 56]	0.5043 02 [0.0047 21]	1.5002 82 [0.0031 91]	1.50056 4 [0.0127 63]
30	0.9983 25 [0.0013 88]	0.9962 79 [0.0068 55]	0.4994 08 [0.0006 68]	0.4986 85 [0.0033 01]	1.5001 46 [0.0019 81]	1.50029 2 [0.0079 323]
40	1.0001 43 [0.0010 09]	1.0003 17 [0.0049 83]	0.5004 03 [0.0005 19]	0.5008 956 [0.0025 62]	1.5011 49 [0.0014 75]	1.50229 7 [0.0059 00]
50	0.9968 58 [0.0008 68]	0.9930 18 [0.0042 88]	0.5005 15 [0.0003 52]	0.5011 45 [0.0017 40]	1.5025 55 [0.0011 97]	1.50510 9 [0.0047 88]
60	1.0013 38 [0.0008 49]	1.0029 74 [0.0041 93]	0.4993 47 [0.0003 67]	0.4985 50 [0.0018 12]	1.4994 04 [0.0009 49]	1.49880 8 [0.0037 98]
70	0.9985 92 [0.0005 66]	0.9968 70 [0.0027 93]	0.5003 20 [0.0002 78]	0.5007 11 [0.0013 72]	1.4995 75 [0.0008 40]	1.49915 0 [0.0033 59]
80	1.0003 48 [0.0005 49]	1.0007 73 [0.0027 11]	0.5002 62 [0.0002 52]	0.5005 83 [0.0012 44]	1.5007 94 [0.0007 45]	1.50158 7 [0.0029 81]

Bayesian Estimation of Hazard and Reverse Hazard Rates for the three parameters Gumbel Distribution under LINEX loss (LLF) function



Graph 2.1(b)



Graph 2.1.(c)

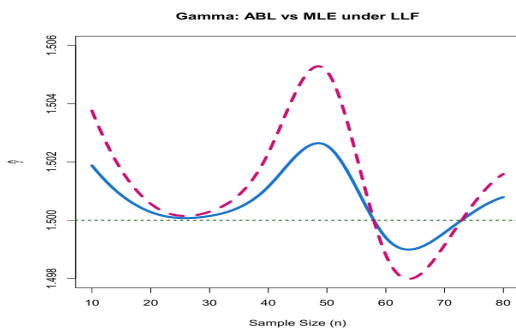


Table 2.1 shows the Approximate Bayes Estimates of the three-parameter Gumbel distribution for the Reverse Hazard Rate Function under LLF. The tables and their corresponding graphs show a similar pattern for the parameters α , β , and γ . It is observed that the Approximate Bayes Estimates are closer to the true parameter values than the Maximum Likelihood Estimates. Furthermore, the Mean Squared Errors decrease with increasing sample size, indicating better estimation performance.

Conclusion

The present study considered the Bayesian estimation of the hazard rate and reverse hazard rate functions for the three-parameter Gumbel distribution under the LINEX loss function. Lindley's approximation was employed to obtain the corresponding Approximate Bayes Estimators, and their performance was compared with that of the Maximum Likelihood Estimators through a simulation study for different sample sizes.

The simulation results show that, for the considered parameter settings, the Approximate Bayes Estimators generally remain closer to the true parameter values and show lower Mean Squared Errors than the corresponding Maximum Likelihood Estimators. Moreover, the Mean Squared Errors generally decrease as the sample size increases,

indicating improved estimation performance with increasing sample size.

Overall, the results establish that Lindley's approximation provides a convenient Bayesian approach for estimating the hazard and reverse hazard rate functions of the three-parameter Gumbel distribution under an asymmetric loss function. The joint consideration of these two reliability characteristics provides useful insight into the reliability behaviour of the model.

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