RESEARCH PAPER

Non – Separable Quadratic- Phase Wavelet Transform in $L^2(R^n)$

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ABSTRACT

With the aim of achieving efficient time-frequency representations of higher-dimensional signals, we introduce the notion of a multidimensional non-separable quadratic-phase wavelet transform by employing the convolution structure of the multidimensional quadratic-phase Fourier transform. Unlike the traditional approach, which associates the classical mother wavelet with the transform kernel, the proposed method employs a convolution-based structure to formulate the analyzing family. This strategy not only reduces computational complexity but also offers significant potential for applications across various scientific and engineering disciplines. The preliminary analysis encompasses the formulation of a novel convolution associated with the multidimensional quadratic-phase Fourier transform. Subsequently, we derive the fundamental properties of the proposed integral transform, including orthogonality and an inversion formula. Additionally, various uncertainty inequalities are obtained corresponding to the pro-posed transform.

Keywords: Quadratic-phase Fourier transform, Wavelet transform, Convolution, Uncertainty principles.

Mathematics Subject Classification: 42A38; 42C20; 11R52; 26D10; SDG9.

How to cite this article: *K. Ravikumar, G. Shanmuga Priya. Non – Separable Quadratic- Phase Wavelet Transform in $L^2(R^n)$. Int J Drug Deliv Technol. 2026;16(88s):1111-1124. DOI: 10.25258/ijddt.16.88s.106*

1. INTRODUCTION

The Fourier transform is a fundamental tool in time-frequency analysis, offering a powerful means to understand functions in terms of their frequency content [1,2]. While the classical Fourier transform has been extensively studied in the one-dimensional setting, many real-world signals, such as images, videos, and geophysical data, are inherently multidimensional. In such cases, the multidimensional Fourier transform (MDFT) arises naturally, providing a rigorous and elegant framework for analyzing functions defined over $\mathbb{R}^n$, where n ≥ 2 [3–5]. For any function the MDFT off is formally defined as

$$\mathcal{F}[f](\zeta) = \frac{1}{2\pi^{n/2}} \int_{\mathbb{R}^n} f(t) e^{-i\zeta^T t}\, dt\,, \tag{1.1}$$

Where $t = (t_1, t_2, \dots, t_n) \in \mathbb{R}^n$, $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n) \in \mathbb{R}^n$, and the superscript ^{T} denotes the transpose operator.

The MDFT provides precise information about the spectral content of a signal and is particularly effective for analyzing stationary signals. However, despite its advantages, the MDFT suffers from inherent limitations, such as, its inability to provide simultaneous localization in both spatial and frequency domains. This lack of time-frequency resolution makes it unsuitable for analyzing non-stationary, localized, or rapidly changing features within signals.

To overcome these limitations, the multidimensional wavelet transform (MDWT) was developed as a natural extension, enabling multiscale and localized analysis [6,7]. Unlike the MDFT, the MDWT decomposes a signal into spatially localized basis functions at various scales, thus offering better resolution for transient or spatially varying phenomena. Its ability to adapt to local features makes it an effective tool for analyzing images, textures, and other multidimensional signals where localized structures are essential. For an admissible mother Wavelet $\psi \in L^2(\mathbb{R}^n)$, the MDWT of any function $f \in L^2(\mathbb{R}^n)$ is defined by

$$(W_\psi f)(a,b) = \frac{1}{\sqrt{a}} \int_{\mathbb{R}^n} f(t)\, \overline{\psi\left(\frac{t-b}{a}\right)}\, dt, \tag{1.2}$$

Where $a \in \mathbb{R}^+$ denotes the scale parameter and $b = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$ the translation parameter. This definition allows the MDWT to capture both spatial and frequency information at multiple resolutions, making it highly effective for the analysis of multidimensional, non-stationary signals.

The MDWT has found successful applications across various fields of science and engineering owing to its

capability to provide localized time-frequency representations [8, 9]. Although the MDWT has proven to be a powerful

tool for analyzing multidimensional signals, it exhibits several notable limitations, including a lack of directional sensitivity, limited adaptability to non-separable structures, and inadequate resolution when dealing with non-stationary or chirp-like signals. These shortcomings restrict its effectiveness in applications that demand high precision in joint time-frequency localization.

To address these challenges, various extensions of MDWT have been developed [10]. Notably, the multidimensional fractional wavelet transform (MDFrWT) enhances analysis by operating in the fractional Fourier domain, enabling rotated time-frequency representations suitable for frequency-modulated signals [11–14]. The multidimensional linear canonical wavelet transform (MDLCWT) further generalizes MDWT by embedding it within the linear canonical transform framework, allowing greater flexibility through scaling, chirping, and shearing [15–19].

Recently, Castro [20] introduced a novel multidimensional quadratic-phase Fourier transform (MDQPFT), which unifies and generalizes several existing time-frequency representations within a single unified framework. The MDQPFT is generated by a collection Q of $n\ X\ n$ real matrices defined as

$$\mathrm{Q} = \{A\ = [a_{ij}], B = [b_{ij}]; det(B) \neq 0\ , C = [c_{ij}], D = [d_{ij}], E = [e_{ij}]\}, \quad (1.3)$$

For $1 \leq i, j \leq n$. mathematically, the MDQPFT of a function $f \in L^2(\mathbb{R}^n)$ with respect to the matrix tuple $Q = \{A, B, C, D, E\}$ is defined as

$$\mathcal{F}_Q[f](\zeta) = \sqrt{\det(B)}\left(\frac{i}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f(t)\, e^{i(\zeta^T A\zeta + \zeta^T Bt \quad t^T Ct + \vec{1}D\zeta + \vec{1}Et)}\, dt, \qquad (1.4)$$

Where B is a symmetric matrix and $\vec{1} = (1,1,\dots,1)^T \in (\mathbb{R}^n)$.

The MDQPFT is characterized by the presence of five real-valued, free $n\ X\ n$ parametric matrices, which provide significantly enhanced flexibility for analyzing signals with complex structures, such as chirp-like or non-stationary components, that are poorly handled by classical Fourier-based techniques. The inclusion of these matrix parameters enables adaptable modulation, chirping, dilation, and shifting, thereby making the MDQPFT a powerful and versatile tool in both theoretical analysis and applied signal processing.

Building upon the strengths of both the MDWT and MDQPFT, we aim to integrate the MDQPFT framework with the MDWT to form a generalized multidimensional quadratic-phase wavelet transform (MDQPWT). Such a fusion is expected to not only encompass existing transforms, such as the MDFT, MDFrWT, and MDLCWT, as special cases, but also to provide deeper insights and enhanced adaptability for analyzing multidimensional non-stationary signals.

The paper is organized into two main parts: In Section 2, we introduce the multidimensional quadratic-phase wavelet transform (MDQPWT) and investigate its fundamental properties. In Section 3, we derive several uncertainty in equalities associated with the proposed MDQPWT.

2. Construction and Properties of the MDQPWT

In this section, we begin by revisiting the fundamentals of the MDQPFT, which serves as the cornerstone for constructing the MDQPWT. We then proceed to formulate the MDQPWT and explore its key properties, including the orthogonality relation, energy preservation, range theorem, and inversion formula.

2.1 Multivariate Quadratic-Phase Fourier Transform.

This subsection presents the essential preliminaries of the MDQPFT.

Definition2.1. The MDQPFT of a function$f \in L^2(\mathbb{R}^n)$ with respect to a matrix tuple $Q = \{A, B, C, D, E\}$, Where B is a symmetric matrix and collection Q of $n\ X\ n$ real matrices, is defined as

$$\mathcal{F}_Q[f](\zeta) = \wedge(B, n) \int_{\mathbb{R}^n} f(t) \mathcal{K}_Q(\zeta, t)\, dt, \qquad (2.1)$$

Where $\mathcal{K}_Q(\zeta, t) = e^{i(\zeta^T A\zeta + \zeta^T Bt + t^T Ct + \vec{1}D\zeta + \vec{1}Et)}$ and $\wedge(B, n) = \sqrt{\det(B)}\left(\frac{i}{2\pi}\right)^{n/2}$.

Remark 2.2.

(i) The MDQPFT (2.1) is non-separable, that is, it cannot be expressed as a tensor product of one-dimensional QPFT kernels.

(ii) We can rewrite the MDQPFT (2.1) in terms of the Fourier transform $\mathcal{F}$ as

$$\mathcal{F}_Q[f](\zeta) = (i)^{n/2}\sqrt{\det(B)}\, e^{i(\zeta^T A\zeta + \vec{1}^T D\zeta)}\mathcal{F}[G](B^T t),$$

Where

$$G(t) = f(t)\, e^{i(t^T Ct + \vec{1}^T Et)}. \tag{2.2}$$

(iii) For any square integrable function f, the inverse of the MDQPFT is given by

$$f(t) = \overline{\wedge(B,n)} \int_{\mathbb{R}^n} \mathcal{F}_Q[f](\zeta)\overline{\mathcal{K}_Q(\zeta,t)}d\zeta. \tag{2.3}$$

(iv) For all functions, $f, g \in L^2(\mathbb{R}^n)$ the MDQPFT preserves the inner product:

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} f(t)\overline{g(t)}\, dt = \int_{\mathbb{R}^n} \mathcal{F}_Q[f](\zeta)\overline{\mathcal{F}_Q[g](\zeta)}\, d\zeta. \tag{2.4}$$

(v) The MDQPFT is an isometry on $L^2(\mathbb{R}^n)$, i.e., it preserves the L^2-norm:

$$\|\mathcal{F}_Q[f]\|_2 = \frac{1}{(2\pi)^{n/2}}\, \|f\|_2\ , \forall\, f \in L^2(\mathbb{R}^n)\,. \tag{2.5}$$

Next, we establish the convolution operation associated with the MDQPFT.

$$\mathcal{F}_Q[f](\zeta)\mathcal{F}_Q[g](\zeta) = [\wedge(B,n)]^2 \int_{\mathbb{R}^n X\, \mathbb{R}^n} f(t)g(t')\, \mathcal{K}_Q(\zeta, t)\, \mathcal{K}_Q(\zeta, t')\, dt\, dt'$$

$$= [\wedge(B,n)]^2 \int_{\mathbb{R}^n X\, \mathbb{R}^n} f(t)g(t') e^{i\left(\zeta^T B(t+t') + (t+t')^T C(t+t') + \vec{1}E(t+t')\right)}$$

$$X\, e^{i(2\zeta^T A\zeta + 2\vec{1}D\zeta)} e^{-i(t^T Ct' + t'^T Ct)}\, dt\, dt'.$$

Setting $z = t + t'$ *and* $t' = z - t$ *gives*

$$\mathcal{F}_Q[f](\zeta)\mathcal{F}_Q[g](\zeta)$$

$$= [\wedge(B,n)]^2 \int_{\mathbb{R}^n X\, \mathbb{R}^n} f(t)g(z - t)\, e^{i(\zeta^T(2A)\zeta\, \vec{1}(2D)\zeta)}\, e^{i(\zeta^T Bz\ \ z^T Cz\ \ \vec{1}Ez)} e^{-i(t^T C(z-t) + (z-t)^T Ct)}\, dt\, dz$$

$$= [\wedge(B,n)]^2 \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(t)g(z-t)\, e^{-i(t^T C(z-t) + (z-t)^T Ct)} dt \right) K_{Q'}(\zeta, z)dz$$

$$= \wedge(B,n) \int_{\mathbb{R}^n} (f \circledast_Q g)(z)\mathcal{K}_{Q'}(\zeta, z)dz$$

$$= \mathcal{F}_{Q'}[(f \circledast_Q g)(z)](\zeta),$$

Where $Q' = (2A, B, C, 2D, E)$ and

$$(f \circledast_Q g)(z) = \wedge(B,n) \int_{\mathbb{R}^n} f(t)g(z-t)e^{-i(t^T C(z-t) + (z-t)^T Ct)}\, dt\,.$$

Hence, the convolution operation associated with the MDQPFT is defined as follows.

Definition 2.3. Given a pair of functions $f, g \in L^2(\mathbb{R}^n)$, the MQPFT convolution with respect to the collection $Q = \{A, B, C, D, E\}$ is denoted by $\circledast_Q$ and is defined as

$$(f \circledast g)(z) = \wedge(B,n) \int_{\mathbb{R}^n} f(t)g(z-t)e^{-i(t^T C(z-t) + (z-t)^T Ct)}dt\,, \tag{2.6}$$

Where $\wedge(B,n)$ is given in (2.1).

Some important properties of the MDQPFT convolution are summarized in Table 1.

TABLE 1. Properties of the MDQPFT convolution $\circledast_Q$.

Property	**Description**
Commutative	$(f \circledast_Q g)(z) = (g \circledast_Q f)(z)$.
Associative	$((f \circledast_Q g) \circledast_Q h)(z) = f \circledast_Q (g \circledast_Q h)(z)$.

Translation	$(f \circledast_Q g)(z-k) = \left(f \circledast_Q e^{t^T Ck + k^T Ct)} \, g(t-k) \right)(z)$.
Reflection	$(f \circledast_Q g)(-z) = \left(f(-t) \circledast_Q \ g(-t)\right)(z)$.
Convolution Theorem	$\mathcal{F}_{Q'}[f \circledast_Q g](\zeta) = \mathcal{F}_Q[f](\zeta)\mathcal{F}_Q[g](\zeta)$, $Q' = (2A, B, C, 2D, E)$.

2.2 The Proposed MDQPWT.

Based on the quadratic-phase convolution (2.6), we introduce the notion of the MDQPWT within the framework of time-frequency analysis. This transform is designed to capture localized frequency content in multiple dimensions by intertwining the merits of MDWT (1.2) and MDQPFT (2.1).

Definition 2.4. Let $f \epsilon L^2(\mathbb{R}^n)$, be a finite-energy signal. The MDQPWT of f, with respect to an analyzing wavelet ψ and a parametric collection $Q = \{A, B, C, D, E\}$, is defined by

$$(W_\psi^Q f)(\eta) = \left(f(t) \circledast_Q D_a \bar{\psi}(-t)\right)(b), \tag{2.7}$$

where $\eta = (a, b) \in \mathbb{R}^+ X \, \mathbb{R}^n$ *and* D is the usual dilation operator

Remark 2.5. The MDQPPWT (2.7) can also be written alternatively as:

$$(W_\psi^Q f)(\eta) = \wedge(\mathrm{B}, \mathrm{n}) \int_{\mathbb{R}^n} f(t) \, \overline{\psi_{\wp}^Q(t)} dt, \tag{2.8}$$

Where $\psi_{\wp}^Q(t)$ is given by

$$\psi_{\wp}^Q(t) = \frac{1}{\sqrt{a}} \psi \left(\frac{t-b}{a}\right) \exp \{i(t^T C(b-t) + (b-t)^T Ct)\}. \tag{2.9}$$

To illustrate the proposed MDQPWT (2.7), we now present a lucid example.

Example 2.6. Consider the 2D-Morlet Wavelet $\psi(t) = e^{i\mu t - |t|^2/2}$, $\mu = (\mu_1, \mu_2) > 0$. then, corresponding to ψ the family $\psi_{\wp}^Q$ of wavelet is given by

$$\psi\left(\frac{t-b}{a}\right) = \exp\left\{-\frac{i(\mu_1 b_1 + \mu_2 b_2)}{a} - \frac{(b_1^2 + b_2^2)}{2a^2}\right\} \exp\left\{-\frac{t_1^2}{2a^2} + \left(\frac{i\mu_1}{a} + \frac{b_1}{a^2}\right) t_1\right\}$$
$$X \exp\left\{-\frac{t_2^2}{2a^2} + \left(\frac{i\mu_2}{a} + \frac{b_2}{a^2}\right) t_2\right\}. \tag{2.10}$$

Then, the MDQPWT of the 2D-Gaussian function $f(t) = f(t_1, t_2) = e^{t_1^2 + t_2^2}$ with respect to Morlet Wavelet $\psi, \mu = (\mu_1, \mu_2) = (1,1)$ and real matrix tuple

$$Q = \left\{ \begin{pmatrix} \frac{11}{5} & 2 \\ -\frac{14}{13} & -\frac{7}{6} \end{pmatrix}, \begin{pmatrix} 1 & \frac{1}{7} \\ \frac{1}{7} & \frac{26}{49} \end{pmatrix}, \begin{pmatrix} 0 & \frac{5}{2} \\ -\frac{5}{2} & 0 \end{pmatrix}, \begin{pmatrix} \frac{15}{39} & -\frac{11}{29} \\ \frac{77}{43} & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{9} & -\frac{3}{29} \\ \frac{7}{3} & 1 \end{pmatrix} \right\}$$

is given by

$$(W_\psi^Q f)(\eta)$$
$$= \frac{i|\det B|^{1/2}}{2\pi\sqrt{a}} \exp\left\{\frac{i(b_1 + b_2)}{a} - \frac{(b_1^2 + b_2^2)}{2a^2}\right\} \int_{\mathbb{R}^n} \exp\{t_1^2 + t_2^2\} \exp\left\{-\frac{t_1^2}{2a^2} + \left(-\frac{i}{a} + \frac{b_1}{a^2}\right) t_1\right\}$$
$$X \exp\left\{-\frac{t_2^2}{2a^2} + \left(-\frac{i}{a} + \frac{b_2}{a^2}\right) t_2\right\} \exp\{-i(t^T c(b-t) + (b-t)^T Ct)\} dt_1 dt_2. \tag{2.11}$$

We now proceed to compute $t^T C(b-t) and \ (b-t)^T Ct$ as follows:

$$t^T c(b-t) = (t_1, t_2) \begin{pmatrix} 0 & \frac{5}{2} \\ -\frac{5}{2} & 0 \end{pmatrix} \begin{pmatrix} b_1 & -t_1 \\ b_2 & -t_2 \end{pmatrix} = -\frac{5}{2} b_1 t_2 + \frac{5}{2} b_2 t_1 \tag{2.12}$$

and

$$(b-t)^T Ct = (b_1 - t_1 \quad b_2 - t_2)\begin{pmatrix} 0 & \frac{5}{2} \\ -\frac{5}{2} & 0 \end{pmatrix}\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \frac{5}{2}b_1t_2 - \frac{5}{2}b_2t_1 \qquad (2.13)$$

Plugging (2.12) and (2.13) in (2.11), we obtain

$$(W_\psi^Q f)(\eta) = \frac{5i}{14\pi\sqrt{a}}\, exp\left\{\frac{i(b_1 + b_2)}{a} - \frac{(b_1^2 + b_2^2)}{2a^2}\right\} X \int_{\mathbb{R}} \exp\left\{-\left(-1 + \frac{1}{2a^2}\right)t_1^2 + \left(-\frac{i}{a} + \frac{b_1}{a^2}\right)t_1\right\} dt_1 \; X \int_{\mathbb{R}} \exp\left\{-\left(-1 + \frac{1}{2a^2}\right)t_2^2 + \left(-\frac{i}{a} + \frac{b_2}{a^2}\right)t_2\right\} dt_2$$

$$= \frac{5i}{14\sqrt{a}\left(-1 + \frac{1}{2a^2}\right)}\, exp\left\{\frac{i(b_1 + b_2)}{a} - \frac{(b_1^2 + b_2^2)}{2a^2}\right\} X \exp\left\{\frac{\left(-\frac{i}{a} + \frac{b_1}{a^2}\right)^2 + \left(-\frac{i}{a} + \frac{b_2}{a^2}\right)^2}{4\left(-1 + \frac{1}{2a^2}\right)}\right\} \qquad (2.14)$$

Next, we establish a fundamental relationship between the MDQPFT given in (2.1) and the proposed MDQPWT defined in (2.7).

Proposition2.7.*Let* $(W_\psi^Q f)(\eta)$ *and* $\mathcal{F}_Q[f]$ *denote the MDQPWT and the MDQPFT of a function* $f \in L^2(\mathbb{R}^n)$*, respectively. Then, the following identity holds:*

$$\mathcal{F}_Q\left[(W_\psi^Q f)(\eta)\right](\zeta) = (2\pi)^n\sqrt{a} \wedge (B, n)\, \mathcal{F}_Q[f](\zeta) e^{i((a\zeta)^T A(a\zeta) + \vec{1}D(a\zeta))} \overline{\mathcal{F}_Q|\psi|(a\zeta)}\,(b) \qquad (2.15)$$

Where

$$\Psi(t) = e^{-i(1+a^2)t^T Ct + (1-a)\vec{1}Et)}\, \psi(t). \qquad (2.16)$$

Proof: Applying the definition of the MDQPFT yields

$$\mathcal{F}_Q\left[\Psi_\wp^Q(t)\right](\zeta) = \frac{\wedge(B, n)}{\sqrt{a}} \int_{\mathbb{R}^n} \psi\left(\frac{t-b}{a}\right) e^{i(t^T C(b-t) + (b-t)^T Ct}\; e^{i(\zeta^T A\zeta + \zeta^T Bt + t^T Ct + \vec{1}D\zeta + \vec{1}Et)}\, dt$$

$$= \sqrt{a} \wedge (B, n) \int_{\mathbb{R}^n} \psi(z)\, e^{i((az+b)^T C(-a) + (-a)^T C(az+b))} X\, e^{i(\zeta^T A\zeta + \zeta^T B(az+b) + (az+b)^T C(az+b) + \vec{1}D\zeta + \vec{1}E(az+b))} dz.$$

$$= \sqrt{a} \wedge (B, n) \int_{\mathbb{R}^n} \psi(z) e^{i((-az)^T C(az) + \zeta^T B(az) + \vec{1}E(az))} e^{i(\zeta^T A\zeta + \zeta^T Bb + b^T Cb + \vec{1}D\zeta + \vec{1}Eb)} dz.$$

$$= \sqrt{a} \wedge (B, n) \int_{\mathbb{R}^n} \psi(z) \mathcal{K}_Q(\zeta, b) e^{i((-az)^T C(az) + \zeta^T B(az) + \vec{1}E(az))} dz.$$

$$= \sqrt{a} \wedge (B, n)\, \mathcal{K}_Q(\zeta, b) \int_{\mathbb{R}^n} \psi(z) e^{i((-a)^T C(az) + \vec{1}E(az) - (a\zeta)^T A(a\zeta) - z^T Cz - \vec{1}D(a\zeta) - \vec{1}Ez)} X\, e^{i((a\zeta)^T A(a\zeta) + (a\zeta)^T Bz + z^T Cz + \vec{1}D(a\zeta) + \vec{1}Ez) dz}$$

$$= \sqrt{a} \wedge (B, n)\, \mathcal{K}_Q(\zeta, b) e^{-i\left((a\zeta)^T A(a\zeta) + \vec{1}D(a\zeta)\right)} \int_{\mathbb{R}^n} \psi(z) e^{-i\left((1+a^2)z^T Cz\;\; (1-a)\vec{1}Ez\right)}\, \mathcal{K}_Q(a\zeta, z)\, dz$$

$$= \sqrt{a} \wedge (B, n)\, \mathcal{K}_Q(\zeta, b) e^{-i\left((a\zeta)^T A(a\zeta) + \vec{1}D(a\zeta)\right)} \int_{\mathbb{R}^n} \psi(z) \mathcal{K}_Q(a\zeta, z)\, dz$$

$$= \sqrt{a} \wedge (B, n)\, \mathcal{K}_Q(\zeta, b) e^{-i\left((a\zeta)^T A(a\zeta) + \vec{1}D(a\zeta)\right)} \mathcal{F}_Q[\psi](a\zeta). \qquad (2.17)$$

where $\psi(t) = e^{-i\left((1+a^2)z^T Ct\;\; (1-a)\vec{1}Ez\right)} \psi(t)$.

Invoking the Parseval's identity for the MDQPFT given in 2.4 and using (2.17), we have

$$(W_\psi^Q f)(\eta) = (2\pi)^n\sqrt{a}|\wedge(\mathrm{B},\mathrm{n})|^2 \int_{\mathbb{R}^n} \mathcal{F}_Q[f](\zeta) e^{i(\mathrm{a}\zeta)^T A(\mathrm{a}\zeta)+\vec{1}D(\mathrm{a}\zeta))} \overline{\mathcal{F}_Q|\psi|(\mathrm{a}\zeta)}\, \overline{\mathcal{K}_\mathrm{Q}(\zeta,\mathrm{b})}\mathrm{d}\zeta$$

$$= \mathcal{F}_Q{}^{-1}\Big[(2\pi)^n\sqrt{a}\wedge(\mathrm{B},\mathrm{n})\mathcal{F}_Q[f](\zeta) e^{i((\mathrm{a}\zeta)^T A(\mathrm{a}\zeta)+\vec{1}D(\mathrm{a}\zeta))}\, \overline{\mathcal{F}_Q|\psi|(\mathrm{a}\zeta)}\Big](\mathrm{b}).$$

Consequently, the above relation can be expressed in the form:

$$\mathcal{F}_Q\big[(W_\psi^Q f)(\eta)\big](\zeta) = (2\pi)^n\sqrt{a}\wedge(\mathrm{B},\mathrm{n})\mathcal{F}_Q[f](\zeta) e^{i(\mathrm{a}\zeta)^T A(\mathrm{a}\zeta)+\vec{1}D(\mathrm{a}\zeta))}\, \overline{\mathcal{F}_Q|\psi|(\mathrm{a}\zeta)}\ (b)$$

This concludes the proof.

□

We now establish key results for the MDQPWT (2.7), including the orthogonality relation, inversion formula, and range theorem. To proceed, we first define the admissibility condition for a function $\psi \in L^2(\mathbb{R}^n)$ via the MDQPFT (2.1)

Definition2.8. A function $\psi \in L^2(\mathbb{R}^n)$ is said to be admissible with respect to the matrix tuple $Q = \{A, B, C, D, E\}$ if

$$\mathcal{B}_\psi = |\det B| \int_{\mathbb{R}+} \frac{\big|\mathcal{F}_Q|\psi|(\mathrm{a}\zeta)\big|^2}{a} da \ < \infty, \qquad (2.18)$$

Where $\psi(\mathrm{z})$ is given by (2.16).

We now derive the orthogonality relation for the proposed transform (2.7), which will be used to show that the MDQPWT defines an isometry from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n X \mathbb{R}^+)$.

Theorem 2.9. *Let $(W_\psi^Q f)(\eta)$ and $(W_\psi^Q g)(\eta)$ denotes the MDQPWTs of f and $g \in L^2(\mathbb{R}^n)$, respectively. Then*

$$\int_{\mathbb{R}^n X \mathbb{R}^+} (W_\psi^Q f)(\eta)\, \overline{(W_\psi^Q g)(\eta)}\, \frac{d\eta}{a^2} = \mathcal{B}_\psi \langle f, g\rangle \qquad (2.19)$$

Where $d\eta = da\, db$ *and* $\mathcal{B}_\psi$ *is defined in* (2.18).

Proof. Proposition 2.7 asserts that, for any pair of functions *f*, $g \in L^2(\mathbb{R}^n)$

$$(W_\psi^Q f)(\eta) = (2\pi)^n\sqrt{a}|\wedge(\mathrm{B},\mathrm{n})|^2 \int_{\mathbb{R}^n} \mathcal{F}_Q[f](\zeta) e^{i((\mathrm{a}\zeta)^T A(\mathrm{a}\zeta)+\vec{1}D(\mathrm{a}\zeta))}\, \overline{\mathcal{F}_Q|\psi|(\mathrm{a}\zeta)}\ \overline{\mathcal{K}_\mathrm{Q}(\zeta,\mathrm{b})}\, \mathrm{d}\zeta. \text{ and}$$

$$(W_\psi^Q g)(\eta) = (2\pi)^n\sqrt{a}|\wedge(\mathrm{B},\mathrm{n})|^2 \int_{\mathbb{R}^n} \mathcal{F}_Q[g](\omega) e^{i((\mathrm{a}\omega)^T A(\mathrm{a}\omega)+\vec{1}D(\mathrm{a}\omega))}\, \overline{\mathcal{F}_Q|\psi|(\mathrm{a}\omega)\ \mathcal{K}_\mathrm{Q}(\omega,\mathrm{b})}\ d\omega.$$

We now proceed as

$$\int_{\mathbb{R}^n X\, \mathbb{R}^+} (W_\psi^Q f)(\eta)\, \overline{(W_\psi^Q f)(\eta)}\, \frac{d\eta}{a^2}$$

$$= (2\pi)^{2n}|\wedge(\mathrm{B},\mathrm{n})|^4 \int_{\mathbb{R}^n X \mathbb{R}^n X \mathbb{R}^n X\, \mathbb{R}^+} e^{i(\mathrm{a}\zeta)^T A(\mathrm{a}\zeta)-(\mathrm{a}\omega)^T A(\mathrm{a}\omega)+\vec{1}D(\mathrm{a}\zeta)-\vec{1}D(\mathrm{a}\omega))} \mathcal{F}_Q[f](\zeta)\overline{\mathcal{F}_Q[g](\omega)}$$

$$\mathrm{X}\ \ \overline{\mathcal{F}_Q|\psi|(\mathrm{a}\zeta)}\, \mathcal{F}_Q|\psi|(\mathrm{a}\omega)\, \overline{\mathcal{K}_\mathrm{Q}(\zeta,\mathrm{b})}\, \mathcal{K}_\mathrm{Q}(\omega,\mathrm{b})\, \frac{d\eta \mathrm{d}\omega \mathrm{d}\zeta}{\mathrm{a}}$$

$$= |\det B|^2 \int_{\mathbb{R}^n\, X\, \mathbb{R}^n\, X\, \mathbb{R}^+} e^{i((\mathrm{a}\zeta)^T A(\mathrm{a}\zeta)-(\mathrm{a}\omega)^T A(\mathrm{a}\omega)+\vec{1}D(\mathrm{a}\zeta)-\vec{1}D(\mathrm{a}\omega))} \mathcal{F}_Q[f](\zeta)\overline{\mathcal{F}_Q[g](\omega)}$$

$$X\, \overline{\mathcal{F}_Q|\psi|(\mathrm{a}\zeta)}\, \mathcal{F}_Q[\psi](\mathrm{a}\omega) \left\{\int_{\mathbb{R}^n} \overline{\mathcal{K}_\mathrm{Q}(\zeta,\mathrm{b})}\, \mathcal{K}_\mathrm{Q}(\omega,\mathrm{b})\, \mathrm{db}\right\} \frac{da\ \mathrm{d}\omega\ \mathrm{d}\zeta}{\mathrm{a}}$$

$$= (2\pi)^n|\det B| \int_{\mathbb{R}^n\, X\, \mathbb{R}^n\, X\, \mathbb{R}^+} e^{i((\mathrm{a}\zeta)^T A(\mathrm{a}\zeta)-(\mathrm{a}\omega)^T A(\mathrm{a}\omega)+\vec{1}D(\mathrm{a}\zeta)-\vec{1}D(\mathrm{a}\omega))} \mathcal{F}_Q[f](\zeta)\overline{\mathcal{F}_Q[g](\omega)}$$

$$X \quad \overline{\mathcal{F}_Q[\psi](\mathrm{a}\zeta)}\, \mathcal{F}_Q[\psi](\mathrm{a}\omega) e^{i(\omega^T A\omega\, -\, (\zeta)^T A\zeta+\vec{1}D\omega\ \ \vec{1}D\zeta)}\, \delta_{(\omega-\zeta)}\, \frac{da\ \mathrm{d}\omega\ \mathrm{d}\zeta}{\mathrm{a}}$$

$$= (2\pi)^n\, |\det B| \int_{\mathbb{R}^n\, X\, \mathbb{R}^+} \mathcal{F}_Q[f](\zeta)\overline{\mathcal{F}_Q[g](\zeta)}\, \mathcal{F}_Q[\psi]|(\mathrm{a}\zeta)|^2\, \frac{da\ \ \mathrm{d}\zeta}{\mathrm{a}}$$

$$= (2\pi)^n \int_{\mathbb{R}^n} \mathcal{F}_Q[f](\zeta)\overline{\mathcal{F}_Q[g](\zeta)} \left\{|\det B| \int_{\mathbb{R}+} \frac{\mathcal{F}_Q[\psi]|(\mathrm{a}\zeta)|^2}{a} da\right\} \mathrm{d}\zeta$$

$$= \mathcal{B}_\psi(2\pi)^n \langle \mathcal{F}_Q[f](\zeta)\, , \mathcal{F}_Q[g](\zeta)\rangle_2$$

$$= \mathcal{B}_\psi\langle f, g\rangle_2.$$

This concludes the proof. □

Remark 2.10. (i).For $f = g$, Theorem 2.9 yields the energy preserving property corresponding to the MDQPWT (2.7):

$$\int_{\mathbb{R}^n X\, \mathbb{R}^+} \left|(W_{\wp}^{Q} f)(\eta)\right|^2 \frac{d\eta}{a^2} = \mathscr{b}_{\psi} \|f\|_2^2 \tag{2.20}$$

(ii). clearly, the operator $(W_{\psi}^{Q} f)$ is a bounded and linear. Furthermore, if $\mathscr{b}_{\psi} = 1$, it defines an isometry from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n X \mathbb{R}^+)$,

Next, we show that any $f \in L^2(\mathbb{R}^n)$ can be recovered from its transformed domain $(W_{\psi}^{Q} f)(\eta)$

Theorem 2.11. *For any $f \in L^2(\mathbb{R}^n)$, MDQPWT $(W_{\psi}^{Q} f)(\eta)$ admits the reconstruction formula*

$$f(t) = \frac{1}{\mathscr{b}_{\psi}} \int_{\mathbb{R}^n X\, \mathbb{R}^+} (W_{\psi}^{Q} f)(\eta)\, \psi_{\wp}^{Q}(t) \frac{d\eta}{a^2}\,, \text{a. e} \tag{2.21}$$

Proof .Using Theorem 2.9, we have

$$\begin{aligned}
\langle f\,, g\rangle &= \frac{1}{\mathscr{b}_{\psi}} \int_{\mathbb{R}^n X\, \mathbb{R}^+} (W_{\psi}^{Q} f)(\eta) \overline{(W_{\psi}^{Q} g)(\eta)} \frac{d\eta}{a^2} \\
&= \frac{1}{\mathscr{b}_{\psi}} \int_{\mathbb{R}^n X\, \mathbb{R}^+} (W_{\psi}^{Q} f)(\eta) \left\{ \int_{\mathbb{R}^n} \overline{g(t)}\, \psi_{\wp}^{Q}(t) dt \right\} \frac{d\eta}{a^2} \\
&= \frac{1}{\mathscr{b}_{\psi}} \int_{\mathbb{R}^n X\, \mathbb{R}^n X\, \mathbb{R}^+} (W_{\psi}^{Q} f)\,(\eta)\, \psi_{\wp}^{Q}(t)\, \overline{g(t)}\, \frac{dt\, d\eta}{a^2} \\
&= \frac{1}{\mathscr{b}_{\psi}} \langle \int_{\mathbb{R}^n X\, \mathbb{R}^+} (W_{\psi}^{Q} f)\,(\eta)\, \psi_{\wp}^{Q}(t) \frac{d\eta}{a^2},\ g(t) \rangle
\end{aligned}$$

As g is arbitrarily in $L^2(\mathbb{R}^n)$, the result follows directly from the inner product properties. □

The following theorems address the pointwise convergence of the reconstruction formula (2.7), assuming that ψ is admissible and $f \in L^2(\mathbb{R}^n)$.

Theorem 2.12. *Let $\psi \in L^2(\mathbb{R}^n)$ be an admissible function. For any $f \in L^2(\mathbb{R}^n)$, define*

$$f_{M,N}(t) = \frac{1}{\mathscr{b}_{\psi}} \int_M^N \int_{\mathbb{R}^n} (W_{\psi}^{Q} f)(\eta) \psi_{\wp}^{Q}(t) \frac{d\eta}{a^2}, \mathrm{N} > \mathrm{M} > 0.$$

Then, the MDQPFT of $f_{M,N}$ satisfies

$$\mathrm{F}_Q[f_{M,N}](\zeta) = \frac{1}{\mathscr{b}_{\psi}} \mathrm{F}_Q[f](\zeta) \left\{ |detB| \int_M^N \frac{\left|\mathrm{F}_Q[\psi](\mathrm{a}\zeta)\right|^2}{a}\, da \right\}$$

Proof. For N > M > 0, define the function

$$f_{M,N}(t) = \frac{1}{\mathscr{b}_{\psi}} \int_M^N \int_{\mathbb{R}^n} (W_{\psi}^{Q} f)(\eta) \psi_{\wp}^{Q}(t) \frac{d\eta}{a^2}$$

Then, as a consequence of the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}
&\int_M^N \int_{\mathbb{R}^n} \left|(W_{\psi}^{Q} f)(\eta)\, \psi_{\wp}^{Q}(t)\right| \frac{d\eta}{a^2} \\
&\leq \int_M^N \left\{ \int_{\mathbb{R}^n} \left|(W_{\psi}^{Q} f)(\eta)\right|^2 db \right\}^{1/2} \left\{ \int_{\mathbb{R}^n} \left|\psi_{\wp}^{Q}(t)\right|^2 db \right\}^{1/2} \frac{da}{a^2} \\
&= \int_M^N \left\{ \int_{\mathbb{R}^n} \left|(W_{\psi}^{Q} f)(\eta)\right|^2 db \right\}^{1/2} \|\psi\|_2 \frac{da}{a^2} \\
&\leq \|\psi\|_2 \left\{ \int_M^N \frac{1}{a^2}\, da \right\}^{1/2} \left\{ \int_M^N \int_{\mathbb{R}^n} \left|(W_{\psi}^{Q} f)(\eta)\right|^2 \frac{d\eta}{a^2} \right\}^{1/2}
\end{aligned}$$

$$\leq \|f\|_2\|\psi\|_2\left(\frac{1}{M}-\frac{1}{N}\right)^{1/2}<\infty.$$

This shows that $f_{M,N}$ is well defined on $\mathbb{R}^n$.

Next, we show that $f_{M,N}$ is uniformly continuous on $\mathbb{R}^n$. For any $t,t'\in\mathbb{R}^n$, we have

$$\left|f_{M,N}(t)-f_{M,N}(t')\right|$$

$$=\left|\frac{1}{\mathcal{B}_\psi}\int_M^N\int_{\mathbb{R}^n}(W_\psi^Q f)(\eta)\left(\psi_{\wp}^Q(t)-\psi_{\wp}^Q(t')\right)\frac{d\eta}{a^2}\right|$$

$$\leq \frac{1}{\mathcal{B}_\psi}\left\{\int_M^N\int_{\mathbb{R}^n}\left|(W_\psi^Q f)(\eta)\right|^2\frac{d\eta}{a^2}\right\}^{1/2}\left\{\int_M^N\int_{\mathbb{R}^n}\left|\psi_{\wp}^Q(t)-\psi_{\wp}^Q(t')\right|^2\frac{d\eta}{a^2}\right\}^{1/2}$$

$$\leq \frac{\|f\|_2}{\sqrt{\mathcal{B}_\psi}}\left\{\int_M^N\int_{\mathbb{R}^n}\left|\left(\psi_{\wp}^Q(t)-\psi_{\wp}^Q(t')\right)\right|^2\frac{d\eta}{a^2}\right\}^{1/2} \qquad (2.22)$$

From (2.22), it is clear that $\left|f_{M,N}(t)-f_{M,N}(t')\right|\to 0\ as\ \ |t-t'|\to 0$. hence $f_{M,N}$ is uniformly continuous on $\mathbb{R}^n$.

Now, for any arbitrary function $g\in L^2(\mathbb{R}^n)$, we have

$$\langle f_{M,N},g\rangle=\int_{\mathbb{R}^n}f_{M,N}(t)\overline{g(t)}\,dt$$

$$=\int_{\mathbb{R}^n}\left\{\frac{1}{\mathcal{B}_\psi}\int_M^N\int_{\mathbb{R}^n}(W_\psi^Q f)(\eta)\psi_{\wp}^Q(t)\frac{d\eta}{a^2}\right\}\overline{g(t)}\,dt$$

$$=\frac{1}{\mathcal{B}_\psi}\int_M^N\int_{\mathbb{R}^n}(W_\psi^Q f)(\eta)\left\{\int_{\mathbb{R}^n}\psi_{\wp}^Q(t)\overline{g(t)}\,dt\right\}\frac{d\eta}{a^2}$$

$$=\frac{1}{\mathcal{B}_\psi}\int_M^N\int_{\mathbb{R}^n}(W_\psi^Q f)(\eta)\,\overline{(W_\psi^Q g)(\eta)}\,\frac{d\eta}{a^2}$$

$$=\frac{|\det B|^2}{\mathcal{B}_\psi}\int_M^N\int_{\mathbb{R}^n\,X\,\mathbb{R}^n\,X\,\mathbb{R}^n}e^{i((\mathrm{a}\zeta)^TA(\mathrm{a}\zeta)-(\mathrm{a}\omega)^TA(\mathrm{a}\omega)+\vec{1}D(\mathrm{a}\zeta)-\vec{1}D(\mathrm{a}\omega))}\,\mathrm{F}_Q[f](\zeta)\overline{\mathrm{F}_Q[g](\omega)}$$

$$\mathrm{X}\ \overline{\mathrm{F}_Q[\psi](\mathrm{a}\zeta)}\,\mathrm{F}_Q[\psi](\mathrm{a}\omega)\,\overline{\mathrm{K}_Q(\zeta,\mathrm{b})}\,\mathrm{K}_Q(\omega,\mathrm{b})\,\frac{d\eta\,\mathrm{d}\omega\,\mathrm{d}\zeta}{\mathrm{a}}$$

$$=\frac{|\det B|^2}{\mathcal{B}_\psi}\int_M^N\int_{\mathbb{R}^n\,X\,\mathbb{R}^n}e^{i((\mathrm{a}\zeta)^TA(\mathrm{a}\zeta)-(\mathrm{a}\omega)^TA(\mathrm{a}\omega)+\vec{1}D(\mathrm{a}\zeta)-\ \vec{1}D(\mathrm{a}\omega))}\,\mathrm{F}_Q[f](\zeta)\overline{\mathrm{F}_Q[g](\omega)}$$

$$\mathrm{X}\ \overline{\mathrm{F}_Q|\psi|(\mathrm{a}\zeta)}\ \mathrm{F}_Q[\psi](\mathrm{a}\omega)\left\{\int_{\mathbb{R}^n}\overline{\mathrm{K}_Q(\zeta,\mathrm{b})}\,\mathrm{K}_Q(\omega,\mathrm{b})\,\mathrm{db}\right\}\frac{da\,\mathrm{d}\omega\,\mathrm{d}\zeta}{\mathrm{a}}$$

$$=\frac{(2\pi)^n|\det B|}{\mathcal{B}_\psi}\int_M^N\int_{\mathbb{R}^n\,X\,\mathbb{R}^n}e^{i((\mathrm{a}\zeta)^TA(\mathrm{a}\zeta)-(\mathrm{a}\omega)^TA(\mathrm{a}\omega)+\vec{1}D(\mathrm{a}\zeta)-\vec{1}D(\mathrm{a}\omega))}\,\mathrm{F}_Q[f](\zeta)\overline{\mathrm{F}_Q[g](\omega)}$$

$$\mathrm{X}\ \overline{\mathrm{F}_Q[\psi](\mathrm{a}\zeta)}\,\mathrm{F}_Q[\psi](\mathrm{a}\omega)e^{i(\zeta^TA\zeta-\omega^TA\omega\ \ \vec{1}D\zeta-\vec{1}D\omega)}\,\delta_{(\zeta-\omega)}\,\frac{da\,\mathrm{d}\omega\,\mathrm{d}\zeta}{\mathrm{a}}$$

$$=\frac{(2\pi)^n|\det B|}{\mathcal{B}_\psi}\int_M^N\int_{\mathbb{R}^n}\mathrm{F}_Q[f](\zeta)\overline{\mathrm{F}_Q[g](\zeta)}\mathrm{F}_Q[\psi]|(\mathrm{a}\zeta)|^2\,\frac{da\,\mathrm{d}\zeta}{a}$$

$$=\frac{(2\pi)^n}{\mathcal{B}_\psi}\int_{\mathbb{R}^n}\mathrm{F}_Q[f](\zeta)\left\{|\det B|\int_M^N\frac{\mathrm{F}_Q[\psi]|(\mathrm{a}\zeta)|^2}{a}\,da\right\}\overline{\mathrm{F}_Q[g](\zeta)}\,\mathrm{d}\zeta.$$

Invoking Parseval's formula for the MDQPFT completes the proof. □

Theorem 2.13. *Suppose that $\psi\in L^2(\mathbb{R}^n)$ is an admissible function. Then for any $f\in L^2(\mathbb{R}^n)$, we have*

$$\lim_{\substack{M\to 0\\N\to\infty}}\left\|f-f_{M,N}\right\|_\infty=0\ and\quad\lim_{\substack{M\to 0\\N\to\infty}}\left\|f-f_{M,N}\right\|_2=0.$$

Proof. Applying Parseval's formula for the MDQPFT along with Theorem 2.12, we obtain

$$\begin{aligned}\|f - f_{M,N}\|_\infty &\le \|f - f_{M,N}\|_1 \\ &= \|\mathcal{F}_Q[f] - \mathcal{F}_Q[f_{M,N}]\|_1 \\ &= \left\| \mathcal{F}_Q[f](\zeta) - \frac{1}{\mathcal{C}_\psi}\mathcal{F}_Q[f](\zeta)\left\{|\det B|\int_M^N \frac{|\mathcal{F}_Q[\psi](a\zeta)|^2}{a}\,\mathrm{da}\right\}\right\|_1 \\ &= \left\| \mathcal{F}_Q[f](\zeta)\left(1 - \frac{|\det B|}{\mathcal{C}_\psi}\int_M^N \frac{|\mathcal{F}_Q[\psi](a\zeta)|^2}{a}\,da\right)\right\|_1 \\ &= \int_{\mathbb{R}^n} |\mathcal{F}_Q[f](\zeta)|\left|1 - \frac{|\det B|}{\mathcal{C}_\psi}\int_M^N \frac{|\mathcal{F}_Q[\psi](a\zeta)|^2}{a}\,da\right| \mathrm{d}\zeta \end{aligned} \tag{2.23}$$

Moreover, since ψ is admissible, we can write

$$\int_M^N \frac{|\mathcal{F}_Q[\psi](a\zeta)|^2}{a}\,\mathrm{da} \le \int_{\mathbb{R}^+} \frac{|\mathcal{F}_Q[\psi](a\zeta)|^2}{a}\,\mathrm{da} = \mathcal{C}_\psi < \infty\,. \tag{2.24}$$

Consequently, from (2.24) we have

$$\lim_{\substack{M\to 0\\ N\to\infty}} \left|1 - \frac{|\det B|}{\mathcal{C}_\psi}\int_M^N \frac{|\mathcal{F}_Q[\psi](a\zeta)|^2}{a}\,\mathrm{da}\right| = 0.$$

Finally, applying dominated convergence theorem to (2.23), we conclude that

$$\lim_{\substack{M\to 0\\ N\to\infty}} \|f - f_{M,N}\|_\infty = 0$$

By adopting the same approach as before, it follows that

$$\lim_{\substack{M\to 0\\ N\to\infty}} \|f - f_{M,N}\|_2 = 0$$

This concludes the proof. □

Finally, we characterize the range of the proposed transform (2.7) and show that $W_\psi^Q(L^2(\mathbb{R}^n))$ forms a reproducing kernel Hilbert space

Theorem 2.14. *A function $f \in L^2(\mathbb{R}^n\ X\ \mathbb{R}^+)$ belongs to the range $W_\psi^Q g(L^2(\mathbb{R}^n))$* if and only if

$$f(\eta') = \frac{1}{\mathcal{C}_\psi}\int_{\mathbb{R}^n\ X\ \mathbb{R}^+} f(\eta)\,\langle \psi_{\wp}^Q(t), \psi_{\wp'}^Q(t)\rangle_2 \frac{d\eta}{a^2}\,, \tag{2.25}$$

Where $\mathcal{C}_\psi$ is given by (2.18).

Proof. Let $f \in W_\psi^Q(L^2(\mathbb{R}^n))$.Then, there exists a function $g \in L^2(\mathbb{R}^n)$ such that $(W_\psi^Q g) = f$.

To verify that f satisfies (2.25), we proceed as follows:

$$\begin{aligned} f(\eta') &= \left(W_\psi^Q g\right)(\eta') \\ &= \int_{\mathbb{R}^n} g(t)\,\overline{\psi_{\wp'}^Q(t)}\,dt \\ &= \frac{1}{\mathcal{C}_\psi}\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n\ X\ \mathbb{R}^+} \left(W_\psi^Q g\right)(\eta)\psi_{\wp'}^Q(t)\frac{d\eta}{a^2}\right\}\overline{\psi_{\wp'}^Q(t)}\,dt \\ &= \frac{1}{\mathcal{C}_\psi}\int_{\mathbb{R}^n\ X\ \mathbb{R}^+} \left(W_\psi^Q g\right)(\eta)\left\{\int_{\mathbb{R}^n} \psi_{\wp}^Q(t)\overline{\psi_{\wp'}^Q(t)}\,dt\right\}\frac{d\eta}{a^2} \\ &= \frac{1}{\mathcal{C}_\psi}\int_{\mathbb{R}^n\ X\ \mathbb{R}^+} f(\eta)\,\langle \psi_{\wp}^Q(t), \psi_{\wp'}^Q(t)\rangle_2 \frac{d\eta}{a^2}\,. \end{aligned}$$

Which evidently verifies our claim conversely, suppose f Satisfies (2.25). To shows that $f \in W_\psi^Q(L^2(\mathbb{R}^n))$, it suffices to find a function $g \in L^2(\mathbb{R}^n)$ such that $(W_\psi^Q f) = f$. we construct such a function g as follows:

Let

$$g(t) = \frac{1}{\mathcal{C}_\psi}\int_{\mathbb{R}^n\ X\ \mathbb{R}^+} f(\eta)\,\psi_{\wp}^Q(t)\frac{d\eta}{a^2}\,.$$

Then, it follows directly that $\|g\|_2 \le \|f\|_2 < \infty$, implying $g \in L^2(\mathbb{R}^n)$. Moreover, by Fubini's theorem, we have

$$
\begin{aligned}
(W_\psi^Q g)(\eta') &= \int_{\mathbb{R}^n} g(t)\overline{\psi_{\wp'}^Q(t)}\, dt \\
&= \frac{1}{\mathcal{B}_\psi}\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n\, X\, \mathbb{R}^+} f(\eta)\psi_\wp^Q(t)\frac{d\eta}{a^2}\right\}\overline{\psi_{\wp'}^Q(t)}\, dt \\
&= \frac{1}{\mathcal{B}_\psi}\int_{\mathbb{R}^n\, X\, \mathbb{R}^+} f(\eta)\,\langle \psi_\wp^Q(t), \psi_{\wp'}^Q(t)\rangle_2 \frac{d\eta}{a^2} \\
&= f(\eta').
\end{aligned}
$$

This concludes the proof. □

Corollary 2.15. *For any admissible wavelet $\psi \in L^2(\mathbb{R}^n)$, the range $W_\psi^Q(L^2(\mathbb{R}^n))$ forms a reproducing kernel Hilbert space embedded in $L^2(\mathbb{R}^n\ X\ \mathbb{R}^+)$, with reproducing kernel*

$$
\mathfrak{K}_\psi^Q(\eta,\eta') = \langle \psi_\wp^Q(t), \psi_{\wp'}^Q(t)\rangle_2. \tag{2.26}
$$

3. Uncertainty Principles for the Free Metaplectic Wavelet Transform

The uncertainty principle forms a cornerstone of harmonic analysis, establishing the inherent limitations in the simultaneous localization of a function and its transform [24]. In mathematical terms, the classical uncertainty principle for the Fourier transform quantifies a trade-off between the spread of a signal in time and in frequency domains [25].Specifically, it asserts that the product of the variances of a function and its Fourier transform has an on zero lower bound. This foundational inequality has since been extended to various domains and analytical frameworks [26–31]. Given the rich structure and enhanced localization properties of the MDQPWT, it is both theoretically and practically compelling to formulate uncertainty principles, such as Heisenberg's and Pitt's inequalities, within the framework of the MDQPWT.

Theorem 3.1. *Let $(W_\psi^Q f)(\eta)$ denote the MDQPWT of a nontrivial function $f \in L^2(\mathbb{R}^n)$. Then, the following uncertainty inequality holds:*

$$
\left\{\int_{\mathbb{R}^n\, X\, \mathbb{R}^+} |b|^2 \left|(W_\psi^Q f)(\eta)\right|^2 \frac{d\eta}{a^2}\right\}^{1/2}\left\{\int_{\mathbb{R}^n} |\zeta|^2 \mathcal{F}_Q[f](\zeta)|^2\, d\zeta\right\}^{1/2} \geq \frac{n\sqrt{\mathcal{B}_\psi}}{2(2\pi)^n \sigma_{\max}(B)}\|f\|_2^2, \tag{3.1}
$$

Where $\sigma_{max}(B)$ is the maximum singular value of the matrix B.

Proof. The classical Heisenberg-Pauli-Weyl uncertainty inequality for any function $f \in L^2(\mathbb{R}^n)$ in the quadratic-phase domain is given by [20]:

$$
\|b\, f(b)\|_2 \quad \left\|\zeta\, \mathcal{F}_Q[f]\right\|_2 \geq \frac{n}{2\,\sigma_{max}(B)}\|\, f(b)\|_2 \tag{3.2}
$$

By treating $(W_\psi^Q f)(\eta)$ as a function of the time variable **b** and applying equation (3.2), we obtain:

$$
\left\|b\,(W_\psi^Q f)(\eta)\right\|_2 \left\|\zeta\, \mathcal{F}_Q[(W_\psi^Q f)(\eta)]\right\|_2 \geq \frac{n}{2\sigma_{max}(B)}\left\|(W_\psi^Q f)(\eta)\right\|_2,
$$

or equivalently,

$$
\begin{aligned}
&\left\{\int_{\mathbb{R}^n} |\, b(W_\psi^Q f)(\eta)|^2 db\right\}^{1/2}\left\{\int_{\mathbb{R}^n} \left|\zeta\, \mathcal{F}_Q[(W_\psi^Q f)(\eta)]\right|^2 d\zeta\right\}^{1/2} \\
&\qquad \geq \frac{n}{2\,\sigma_{max}(B)}\left\{\int_{\mathbb{R}^n} \left|(W_\psi^Q f)(\eta)\right|^2 db\right\}^{1/2}
\end{aligned} \tag{3.3}
$$

Integrating (3.3) with respect to the measure da/a^2, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^+}\left\{\int_{\mathbb{R}^n} |b|^2 \left|(W_\psi^Q f)(\eta)\right|^2 db\right\}^{1/2}\left\{\int_{\mathbb{R}^n} |\zeta\,|^2 \left|\mathcal{F}_Q[(W_\psi^Q f)(\eta)](\zeta)\right|^2 d\zeta\right\}^{1/2}\frac{da}{a^2} \\
&\qquad \geq \frac{n}{2\,\sigma_{max}(B)}\left\{\int_{\mathbb{R}^n\, X\, \mathbb{R}^+} \left|(W_\psi^Q f)(\eta)\right|^2 \frac{d\eta}{a^2}\right\}
\end{aligned} \tag{3.4}
$$

As a consequence of the Cauchy-Schwarz inequality, Fubini's theorem, and (2.20), equation (3.4) can be expressed as:

$$
\left\{\int_{\mathbb{R}^n\, X\, \mathbb{R}^+} |b|^2 \left|(W_\psi^Q f)(\eta)\right|^2 \frac{d\eta}{a^2}\right\}^{1/2}\left\{\int_{\mathbb{R}^n\, X\, \mathbb{R}^+} |\zeta\,|^2 \left|\mathcal{F}_Q[(W_\psi^Q f)(\eta)](\zeta)\right|^2 \frac{d\zeta\, da}{a^2}\right\}^{1/2} \geq \frac{n\mathcal{B}_\psi}{2\,\sigma_{max}(B)}\|f\|_2^2.
$$

Invoking Proposition 2.7, we rewrite the above inequality as:

$$\left\{\int_{\mathbb{R}^n\ X\ \mathbb{R}^+}|b|^2\left|\left(W_\psi^Q f\right)(\eta)\right|^2\frac{d\eta}{a^2}\right\}^{1/2}\left\{(2\pi)^n|\det B|\int_{\mathbb{R}^n\ X\ \mathbb{R}^+}|\zeta|^2\left|\mathrm{F}_Q[f](\zeta)\mathrm{F}_Q[\psi](\mathrm{a}\zeta)\right|^2\frac{d\zeta\,da}{a^2}\right\}^{1/2}\geq\frac{n\mathcal{B}_\psi}{2\,\sigma_{max}(B)}\|f\|_2^2,$$

or equivalently,

$$\left\{\int_{\mathbb{R}^n\ X\ \mathbb{R}^+}|b|^2\left|\left(W_\psi^Q f\right)(\eta)\right|^2\frac{d\eta}{a^2}\right\}^{1/2}\left\{\int_{\mathbb{R}^n}|\zeta|^2\left|\mathrm{F}_Q[f](\zeta)\right|^2\left(|\det B|\int_{\mathbb{R}^+}\frac{\left|\mathrm{F}_Q[\psi](\mathrm{a}\zeta)\right|^2}{a}da\right)d\zeta\right\}^{1/2}\geq\frac{n\mathcal{B}_\psi}{2\,(2\pi)^n\,\sigma_{max}(B)}\|f\|_2^2,$$

Finally, applying (2.18) yields the desired result. □

While the classical Heisenberg uncertainty principle focuses on the spread (or variance) of a function and its transform, Nazarov's uncertainty principle offers a more geometric perspective by considering the support of the function and its Fourier transform [32].Mathematically, if $S_1 and\ S_2$ are two finite measurable subset of $\mathbb{R}^n$ and $f\in L^2(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n}|f(t)|^2\,dt\ \leq Ke^{K|S_1||S_2|}\left(\int_{\mathbb{R}^n\setminus S_1}|f(t)|^2\ dt+\int_{\mathbb{R}^n\setminus S_2}|\mathrm{F}[f](\zeta)|^2\,d\zeta\right),k>0\,, \qquad (3.5)$$

Where $|S_1|and\ |S_2|$ denotes the Lebesgue measures of $S_1\ and\ S_2$respectively. In essence, it is impossible for a non-trivial function and its Fourier transform to be simultaneously too small outside two compact (or small measure) sets. In this regard, we have the following proposition for the MDQPFT.

Proposition3.2. *Let $\mathrm{F}_Q[f]$ denotes the MDQPFT of a function $f\in L^2(\mathbb{R}^n)$, and let S_1,S_2 be two finite measurable subsets of $\mathbb{R}^n$.Then the following inequality holds:*

$$\int_{\mathbb{R}^n}|f(t)|^2\,dt\ \leq Ke^{K|S_1||S_2|}\left(\int_{\mathbb{R}^n\setminus S_1}|f(t)|^2\,dt+\int_{\mathbb{R}^n\setminus(S_2\setminus|B|)}\left|\mathrm{F}_Q[f](\zeta)\right|^2d\zeta\right), \qquad (3.6)$$

Where $|S_1|and\ |S_2|$ denote the measrable $S_1\ and\ S_2$.

Proof. Invoking Nazarov's inequality (3.5) for the function G (**t**), we obtain

$$\int_{\mathbb{R}^n}|G(t)|^2\,dt\leq Ke^{K|S_1||S_2|}\left(\int_{\mathbb{R}^n\setminus S_1}|G(t)|^2\,dt+\int_{\mathbb{R}^n\setminus S_2}|\mathrm{F}[G](\zeta)|^2\,d\zeta\right),k>0 \qquad (3.7)$$

For the function G (t) = f(t) $e^{i(t^TCt+\vec{1}Et)}$ as defined in (2.2) ,we have

$$\mathrm{F}[G](\zeta)\ =\frac{1}{i^{n/2}\sqrt{\det B}}e^{-i(\zeta^T\tilde{A}\zeta+\vec{1}\tilde{D}\zeta)}\mathrm{F}_Q[f](B^{-1}\zeta), \qquad (3.8)$$

Using (3.8) in (3.7), we obtain

$$\int_{\mathbb{R}^n}|f(t)|^2\,dt\ \leq Ke^{K|S_1||S_2|}\left(\int_{\mathbb{R}^n\setminus S_1}|f(t)|^2\,dt+\frac{1}{|B|}\int_{\mathbb{R}^n\setminus(S_2\setminus|B|)}\left|\mathrm{F}_Q[f](B^{-1}\zeta)\right|^2d\zeta\right).$$

Equivalently,

$$\int_{\mathbb{R}^n}|f(t)|^2\,dt\ \leq Ke^{K|S_1||S_2|}\left(\int_{\mathbb{R}^n\setminus S_1}|f(t)|^2\,dt+\int_{\mathbb{R}^n\setminus(S_2\setminus|B|)}\left|\mathrm{F}_Q[f](\zeta)\right|^2d\zeta\right)$$

This completes the proof of Proposition 3.2. □

We now proceed to formulate a Nazarov-type uncertainty inequality for the MDQPWT.

Theorem 3.3. *Let $(W_\psi^Q f)(\eta)$ denotes the MDQPWT defined in (2.7), and let S_1,S_2 be two finite measurable subsets of $\mathbb{R}^n$. Then the following inequality holds:*

$$\int_{\mathbb{R}^n\ X\ \mathbb{R}^+}\left|\left(W_\psi^Q f\right)(\eta)\right|^2\frac{d\eta}{a^2}\leq Ke^{K|S_1||S_2|}\begin{pmatrix}\int_{(\mathbb{R}^n\setminus S_1)\ X\ \mathbb{R}^+}\left|\left(W_\psi^Q f\right)(\eta)\right|^2\frac{d\eta}{a^2}\\+(2\pi)^n\mathcal{B}_\psi\int_{\mathbb{R}^n\setminus(S_2\setminus|B|)\ X\ \mathbb{R}^+}\left|\mathrm{F}_Q[f](\zeta)\right|^2d\zeta\end{pmatrix},K>0. \qquad (3.9)$$

Proof. By substituting the function f with its MDQPWT $\left(W_\psi^Q f\right)(\eta)$ in (3.6), we obtain

$$\int_{\mathbb{R}^n}\left|\left(W_\psi^Q f\right)(\eta)\right|^2db\ \leq\ Ke^{K|S_1||S_2|}\left(\int_{(\mathbb{R}^n\setminus S_1)}\left|\left(W_\psi^Q f\right)(\eta)\right|^2db+\int_{\mathbb{R}^n\setminus(S_2\setminus|B|)}\left|\mathrm{F}_Q\left[\left(W_\psi^Q f\right)(\eta)\right](\zeta)\right|^2d\zeta\right) \qquad (3.10)$$

Integrating (3.10) with respect to the measure da/a^2, we obtain

$$\int_{\mathbb{R}^n\ X\ \mathbb{R}^+}\left|\left(W_\psi^Q f\right)(\eta)\right|^2\frac{d\eta}{a^2}$$

$$\leq Ke^{K|S_1||S_2|}\left(\int_{(\mathbb{R}^n\setminus S_1)\,X\,\mathbb{R}^+}\left|(W_\psi^Q f)(\eta)\right|^2\frac{d\eta}{a^2}+\int_{\mathbb{R}^n\setminus(S_2\setminus|B|)\,X\,\mathbb{R}^+}\left|\mathrm{F}_Q\left[(W_\psi^Q f)(\eta)\right](\zeta)\right|^2\frac{d\zeta da}{a^2}\right). \quad (3.11)$$

Applying Proposition 2.7, equation (3.11) can now be rewritten as:

$$\int_{\mathbb{R}^n\,X\,\mathbb{R}^+}\left|(W_\psi^Q f)(\eta)\right|^2\frac{d\eta}{a^2}$$

$$\leq Ke^{K|S_1||S_2|}\left(\int_{(\mathbb{R}^n\setminus S_1)\,X\,\mathbb{R}^+}\left|(W_\psi^Q f)(\eta)\right|^2\frac{d\eta}{a^2}+(2\pi)^n\int_{(\mathbb{R}^n\setminus(S_2\setminus|B|))\,X\,\mathbb{R}^+}\left|\mathrm{F}_Q[f](\zeta)\right|^2 X\left(|\det B|\int_{\mathbb{R}^+}\frac{\left|\mathrm{F}_Q[\psi]a\zeta\right|^2}{a}da\right)d\zeta\right) \quad (3.12)$$

Finally, Invoking equation (2.18) leads to the desired result. □

We now briefly recall the definition of the Schwartz space $I(\mathbb{R}^n)$ within the $L^2(\mathbb{R}^n)$ frame-work. It is defined as

$$I(\mathbb{R}^n)=\left\{f\in C^\infty(\mathbb{R}^n):\sup_{t\in\mathbb{R}^n}t^\beta D^\gamma f(t)<\infty\right\}, \quad (3.13)$$

Where $C^\infty(\mathbb{R}^n)$ denotes the space of infinitely differentiable functions, D represents the standard partial differential operator, and α,β are multi-indices with non-negative integer components.

Pitt's inequality in the Fourier domain establishes a fundamental relationship between a sufficiently smooth function and its Fourier transforms [33]. For every $f\in I(\mathbb{R}^n)\subseteq L^2(\mathbb{R}^n)$, the inequality is given by

$$\int_{\mathbb{R}^n}|\zeta|^{-\propto}|\,\mathrm{F}[f](\zeta)|^2d\zeta\leq\wp_\alpha\int_{\mathbb{R}^n}|\mathrm{t}|^\alpha|\mathrm{f}(\mathrm{t})|^2\mathrm{dt},0\leq\alpha<n, \quad (3.14)$$

Where

$$\wp_\alpha=\pi^\alpha\left[\Gamma\left(\frac{n-\alpha}{4}\right)/\Gamma\left(\frac{n+\alpha}{4}\right)\right]^2, \quad (3.15)$$

and $\Gamma(\cdot)$ denotes the well-known Euler's gamma function.

In the following proposition, we aim to derive an analogue of logarithmic uncertainty inequality for the non-separable MDQPFT.

Proposition 3.4. *For any $f\in I(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}^n}\ln|t||f(t)|^2\,dt+\int_{\mathbb{R}^n}\ln|\zeta|\left|\mathrm{F}_Q[f](\zeta)\right|^2d\zeta\geq\left[\frac{\Gamma'(\mathrm{n}/4)}{\Gamma(\mathrm{n}/4)}-\ln\pi-\frac{\ln[\mathrm{B}]}{(2\pi)^{\mathrm{n}}}\right]\int_{\mathbb{R}^n}|\mathrm{f}(\mathrm{t})|^2\mathrm{dt} \quad (3.16)$$

Proof. For the function G (t) = f (t) $e^{i(t^TCt+\vec{1}Et)}$ as defined in (2.2), we have

$$\mathrm{F}[G](\zeta)=\frac{1}{i^{n/2}\sqrt{\det B}}e^{-i(\zeta^T+\tilde{A}\zeta+\vec{1}\tilde{D}\zeta)}\mathrm{F}_Q[f](B^{-1}\zeta), \quad (3.17)$$

Where $\tilde{A}=(B^{-1})^TAB^{-1}$ and $\tilde{D}=DB^{-1}$. Moreover, we have

$$|\,\mathrm{F}[G](\zeta)|=\frac{1}{\sqrt{|detB|}}\left|\mathrm{F}_Q[f](B^{-1}\zeta)\right|,\text{ and }|G(t)|=|f(t)| \quad (3.18)$$

Applying inequality (3.14) to the function G (**t**), we obtain:

$$\int_{\mathbb{R}^n}|\zeta|^{-\propto}|\mathrm{F}[G](\zeta)|^2d\zeta\leq\wp_\alpha\int_{\mathbb{R}^n}|\mathrm{t}|^\alpha|\mathrm{G}(\mathrm{t})|^2\mathrm{dt},\qquad 0\leq\alpha<n \quad (3.19)$$

Using equation (3.18) in (3.19), we have

$$\frac{1}{|\det(B)|}\int_{\mathbb{R}^n}|\zeta|^{-\propto}\left|\mathrm{F}_Q[f](\mathrm{B}^{-1}\zeta)\right|^2d\zeta\leq\wp_\alpha\int_{\mathbb{R}^n}|\mathrm{t}|^\alpha|\mathrm{f}(\mathrm{t})|^2\mathrm{dt}\ ,0\leq\alpha<n\,.$$

Equivalently, we can write

$$|\mathrm{B}|^{-\alpha}\int_{\mathbb{R}^n}|\zeta|^{-\propto}\left|\mathrm{F}_Q[f](\zeta)\right|^2d\zeta\leq\wp_\alpha\int_{\mathbb{R}^n}|\mathrm{t}|^\alpha|\mathrm{f}(\mathrm{t})|^2\mathrm{dt}\ ,0\leq\alpha<n. \quad (3.20)$$

Inequality (3.20) is the Pitt's inequality for the MDQPFT.

Additionally, for every $0\leq\alpha<n$, we define:

$$\mathrm{M}(\alpha)=|B|^{-\alpha}\int_{\mathbb{R}^n}|\zeta|^{-\propto}\left|\mathrm{F}_Q[f](\zeta)\right|^2d\zeta-\wp_\alpha\int_{\mathbb{R}^n}|\mathrm{t}|^\alpha|\mathrm{f}(\mathrm{t})|^2\mathrm{dt}. \quad (3.21)$$

Differentiating (3.21) yields

$$\mathrm{M}'(\alpha)=-|B|^{-\alpha}\ln|B|\int_{\mathbb{R}^n}|\zeta|^{-\propto}\left|\mathrm{F}_Q[f](\zeta)\right|^2d\zeta-|B|^{-\alpha}\int_{\mathbb{R}^n}|\zeta|^{-\propto}\ln|\zeta|\left|\mathrm{F}_Q[f](\zeta)\right|^2d\zeta$$

$$-\wp_\alpha \int_{\mathbb{R}^n} |t|^\alpha \ln|t|\, |f(t)|^2 dt - \wp'_\alpha \int_{\mathbb{R}^n} |t|^\alpha |f(t)|^2 dt \ .$$

Where

$$\wp'_\alpha = -\frac{\pi^\alpha}{2}\left\{\frac{\Gamma^2\left(\frac{n+a}{4}\right)\Gamma\left(\frac{n-a}{4}\right)\Gamma'\left(\frac{n-a}{4}\right)+\Gamma^2\left(\frac{n-a}{4}\right)\Gamma\left(\frac{n+a}{4}\right)\Gamma'\left(\frac{n+a}{4}\right)}{\Gamma^4\left(\frac{n+a}{4}\right)}\right\}+\pi^\alpha \ln\pi\left\{\frac{\Gamma^2\left(\frac{n-a}{4}\right)}{\Gamma^2\left(\frac{n+a}{4}\right)}\right\}.$$

Putting $\alpha = 0$ into the above equation, we get

$$\wp'_0 = \left[\ln\pi - \frac{\Gamma'(n/4)}{\Gamma(n/4)}\right]. \qquad (3.22)$$

By virtue of (3.15), we have, $\mathrm{M}(\alpha) \le 0$ for all $\alpha \in [0, n)$, with equality at $\alpha = 0$, i.e., $\mathrm{M}(0) = 0$. therefore, for any $h > 0$, it follows that $\mathrm{M}'(0+h) \le 0\ as\ h \to 0. Hence,$

$$-\ln|B| \int_{\mathbb{R}^n} \left|\mathrm{F}_Q[f](\zeta)\right|^2 d\zeta - \int_{\mathbb{R}^n} \ln|\zeta| \left|\mathrm{F}_Q[f](\zeta)\right|^2 d\zeta - \wp_0 \int_{\mathbb{R}^n} \ln|t|\, |f(t)|^2 dt \quad - \wp'_0 \int_{\mathbb{R}^n} |f(t)|^2 dt \le 0 \ .$$

Finally, implementing the Parseval's formula (2.4) yields the desired result. □

Next, we establish the logarithmic uncertainty inequality for the MDQPWT (3.20).

Theorem 3.5. *Let* $(W_\psi^Q f)(\eta)$ *denote the MDQPWT .Then, the following logarithmic inequality holds:*

$$\int_{\mathbb{R}^n\, x\, \mathbb{R}^+} \ln|b| \left|(W_\psi^Q f)(\eta)\right|^2 \frac{d\eta}{a^2} + (2\pi)^n \mathcal{B}_\psi \int_{\mathbb{R}^n} \ln|\zeta| \left|\mathrm{F}_Q[f](\zeta)\right|^2 d\zeta \ge \mathcal{B}_\psi \left[\frac{\Gamma'(n/4)}{\Gamma(n/4)} - \ln\pi - \frac{\ln|B|}{(2\pi)^n}\right] \|f\|_2^2. \quad (3.23)$$

Proof. By virtue of the logarithmic–type uncertainty principle in the MQPFT domain given by equation (3.16), we identify$(W_\psi^Q f)(\eta)$ as a function of the time variable **t** and invoke this relation to obtain:

$$\int_{\mathbb{R}^n} \ln|b| \left|(W_\psi^Q f)(\eta)\right|^2 db + \int_{\mathbb{R}^n} \ln|\zeta| \left|\mathrm{F}_Q(W_\psi^Q f)(\eta)(\zeta)\right|^2 d\zeta \ge \left[\frac{\Gamma'(n/4)}{\Gamma(n/4)} - \ln\pi - \frac{\ln|B|}{(2\pi)^n}\right] \int_{\mathbb{R}^n} \left|(W_\psi^Q f)(\eta)\right|^2 db.$$

Integrating with respect to da/a^2 , we obtain

$$\int_{\mathbb{R}^+\, x\, \mathbb{R}^n} \ln|b| \left|(W_\psi^Q f)(\eta)\right|^2 \frac{d\eta}{a^2} + \int_{\mathbb{R}^+ x \mathbb{R}^n} \ln|\zeta| \left|\mathrm{F}_Q(W_\psi^Q f)(\eta)(\zeta)\right|^2 \frac{d\zeta\, da}{a^2} \ge \left[\frac{\Gamma'(n/4)}{\Gamma(n/4)} - \ln\pi - \frac{\ln|B|}{(2\pi)^n}\right] \int_{\mathbb{R}^+\, x\, \mathbb{R}^n} \left|(W_\psi^Q f)(\eta)\right|^2 \frac{d\eta}{a^2}. \qquad (3.24)$$

Now, by applying Proposition 2.7, equation (3.24) can be rewritten as:

$$\int_{\mathbb{R}^+\, x\, \mathbb{R}^n} \ln|b| \left|(W_\psi^Q f)(\eta)\right|^2 \frac{d\eta}{a^2} + (2\pi)^n \int_{\mathbb{R}^n} \ln|\zeta| \left|\mathrm{F}_Q[f](\zeta)\right|^2 \left(|\det \mathrm{B}| \int_{\mathbb{R}^+} \frac{\left|\mathrm{F}_Q[\psi](a\zeta)\right|^2}{a} da\right) d\zeta \ge \left[\frac{\Gamma'(n/4)}{\Gamma(n/4)} - \ln\pi - \frac{\ln|B|}{(2\pi)^n}\right] \int_{\mathbb{R}^+\, x\, \mathbb{R}^n} \left|(W_\psi^Q f)(\eta)\right|^2 \frac{d\eta}{a^2}. \qquad (3.25)$$

Finally, substituting equation (2.18) into (3.25), we obtain the desired result. □

Competing interests: The authors declare that they have no competing interests.

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