RESEARCH PAPER

A STUDY ON THE PROPERTIES OF CONVEX TRANSFORM ORDER WITH SOME RELATED PARTIAL ORDERS USING FUZZY RANDOM VARIABLES AND ITS APPLICATION

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Abstract

In this paper we have discussed some properties of convex transform order with other related partial orders using fuzzy random variables. We have introduced some new definitions, proposed the relation between the partial orders and proved how they are implied from the other using fuzzy as a tool. And, we have also included one application of fuzzy convex transform order in Poisson process.

Keywords:

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1.Introduction:

Stochastic orders pays a way to relate the location or magnitude of random variables to check if one tends to take bigger values than the other. They also compare vectors, probability distributions rather than just comparing values like mean and variance (Stoyan [11]). They have been useful in reliability theory, economics, insurance etc. For good review, refer Shaked and Shanthikumar [10]. A Fuzzy Mathematics which was introduced by Zadeh [14] which is a concept of fuzzy sets, offers a mathematical framework for modeling and analyzing systems characterized by vagueness and uncertainity in a way that crisp mathematics fails.

Here we are going to study about convex transform order of fuzzy random variables. Van Zwet [12] introduced the convex transform order between two distribution functions F and G. Among the various stochastic orders, convex transform order (CTO) is widely used for analyzing variability and aging phenomena or tail behaviorism by analyzing whether one distribution's can be showed as convex transformation of other's. This partial order is useful in the reliability engineering, risk management and actuarial science. Along with CTO, we study some related orders like star, super additive, TTT transform, decreasing mean residual life order using FRVs. We have shown how they are related to CTO other using fuzzy logic.

$$\tilde{X} \leq_{fct} \tilde{Y} \quad \Rightarrow \quad \tilde{X} \leq_{f*} \tilde{Y} \quad \Rightarrow \quad \tilde{X} \leq_{fsu} \tilde{Y}$$
$$\Downarrow$$
$$\tilde{X} \leq_{f-dmrl} \tilde{Y} \Rightarrow \quad \tilde{X} \leq_{f-nbue} \tilde{Y}$$

The concept of fuzzy random variable (FRV) was introduced by Kwakernaak and Puri [4], [5] and Ralscue [7]. A fuzzy random variable is a measurable function from a probability space to a collection of fuzzy sets. It is viewed as the imprecise observation of the outcomes in a random experiment. In some practical problems, data is often incomplete. To solve this, we have the extention concept as fuzzy framework. This is more applicable in imprecise conditions.

In section 2 we have given some preliminaries related to this paper, in section 3, we discussed the relation of the fuzzy convex transform orderings to the other partial orders and given a counter example and in section 4 we have discussed application of CTO using FRVs in Poisson process through an example.

2. Preliminaries:

Definition 2.1: Fuzzy set

We consider X be the universal set. Then a fuzzy set $\tilde{A} = \{(x, \mu_A(x))/\ x \in X\}$ is defined by its membership function $\mu_A: X \to [0,1]$.

Definition 2.2: $\alpha -$ cut of a fuzzy set

For each $0 \leq \alpha \leq 1$, the $\alpha -$ cut of a fuzzy set $\tilde{A}$ is denoted by $\tilde{A}_\alpha = \{\, x \in X;\ \mu_{\tilde{A}}(x) \geq \alpha\}$

Definition 2.3: Fuzzy random variable (FRV)

$\tilde{X}(\omega)$ is a fuzzy random variable if and only if $\tilde{X}_\alpha(\omega) = \left[\tilde{X}_\alpha^L(\omega), X_\alpha^U(\omega)\right]$ where $\tilde{X}_\alpha^L(\omega)$ and $X_\alpha^U(\omega)$ are both random variables for each $\alpha \in [0,1]$ and $X = \bigcup_{\alpha\in[0,1]} \alpha\, \tilde{X}_\alpha(\omega)$.

Definition 2.4: Uniform FRV

$\tilde{X}(\omega)$ is a uniform [0,1] FRV if and only if $\tilde{X}_{U_\alpha}(\omega) = \left[\tilde{X}_{U_\alpha}^{\ L}(\omega), \tilde{X}_{U_\alpha}^{\ U}(\omega)\right]$ where $\tilde{X}_{U_\alpha}^{\ L}(\omega)$ and $\tilde{X}_{U_\alpha}^{\ U}(\omega)$ are both random variables for each $\alpha \in [0,1]$ and $\tilde{X} = \bigcup_{\alpha\in[0,1]} \alpha\, \tilde{X}_{U_\alpha}(\omega)$.

Definition 2.5: $\alpha -$ cut of the distribution function F of Fuzzy random variable

The $\alpha -$ cut of the distribution function F of Fuzzy random variable $\tilde{X}$ is defined by

$$D_\alpha = \left[Min\left(\min_{\alpha \leq \beta} F(x_\beta^L), \min_{\alpha \leq \beta} F(x_\beta^U)\right), Max\left(\max_{\alpha \leq \beta} F(x_\beta^L), \max_{\alpha \leq \beta} F(x_\beta^U)\right)\right]$$

Definition 2.6: Convex transform order (CTO)

Suppose that the support of X is an interval. We can tell that X is lesser than or equal to Y in the CT order represented as, $X \leq_{ct} Y$, if $G^{-1}F\ (x)$ is convex in x on the support of F.

Definition 2.7: Condition satisfying convex transform order (CT)

We can tell that X is lesser than or equal to Y in the CT order represented as, $X \leq_{ct} Y$ if and only if, $\frac{f\left(F^{-1}(u)\right)}{g\left(G^{-1}(u)\right)}$ is increasing in u ∈ [0,1].

Where f and g are the densities associated with F and G respectively.

Definition 2.8: Star shaped order (SSO)

A function $\acute{\phi}$: [0,∞) → (0,∞] which satisfies $\acute{\phi}(0) = 0$, is called starshaped if $\frac{\acute{\phi}(x)}{x}$ is increasing in x on (0,∞).i.e., $\frac{b}{\infty} = 0\ for\ b > 0$. It can be noted that such type of function is increasing. Also note that every increasing convex function $\acute{\phi}$ on [0,∞), such that $\acute{\phi}(0) = 0$, is starshaped.

Let X and Y be two random variables. We can tell that X is lesser than or equal to Y in the SS order as represented as $X \leq_{ss} Y$, then $E[\acute{\phi}(x)] \leq E[\acute{\phi}(y)]$, for all starshaped functions $\acute{\phi}: R \to R$.

Definition 2.9: Star order (∗)

Suppose that the support of X is an interval. We can tell that X is lesser than or equal to Y in the Star order represented as, $X \leq_* Y$, if $G^{-1}F\ (x)$ is starshaped in x.

i.e., if $\frac{G^{-1}F\ (x)}{x}$ increases in $x \geq 0$.

Definition 2.10: Super additive order (SUO)

Suppose that the support of X is an interval. We can tell that X is lesser than or equal to Y in the SU order as represented as, $X \leq_{su} Y$, if $G^{-1}F\ (x)$ is super additive in x.

i.e., if $G^{-1}F\ (x+y) \geq G^{-1}F\ (x) + G^{-1}F\ (y)$ for all $x \geq 0$ and $y \geq 0$.

Definition 2.11: TTT (Total Time on Test) transform

Let X and Y be considered as positive random variables which has the distribution functions F and G, suppose 0 is the left endpoint of the support of F.

The (TTT) Total time on test, transform can be defined for F and G as

$$H_{F^{-1}}(u) = \int_0^{F^{-1}(u)} \bar{F}(x)\, dx, 0 \leq u \leq 1$$

$$H_G^{-1}(u) = \int_0^{G^{-1}(u)} \bar{G}(x)\, dx, 0 \leq u \leq 1$$

where $\bar{F} = 1 - F$, $\bar{G} = 1 - G$ the survival functions connected with F and G. The inverse, H_F and H_G of the TTT transform is a distribution function.

Definition 2.12: TTT (Total Time on Test) Random variable

Let X and Y be considered as positive random variables which has the distribution functions F and G, suppose 0 is the left endpoint of the support of F.

If X and Y has the distribution function F and G, then X_{ttt} and Y_{ttt} is called the observed total time on test random variable that has the distributions H_F *and* H_G respectively

The density function h_F of the inverse of the TTT transform H_F that is connected with F satisfies

$$h_F\left(H_F^{-1}(u)\right) = \frac{f\left(F^{-1}(u)\right)}{1-u},\ 0 \leq u \leq 1$$

where f is the density function associated with F. Similarly, then the density h_G, of the inverse of the TTT transform H_G that is associated with G satisfies

$$h_G\left(H_G^{-1}(u)\right) = \frac{g\left(G^{-1}(u)\right)}{1-u},\ 0 \leq u \leq 1$$

where g is the density function associated with G.

Definition 2.13: DMRL (Decreasing Mean Residual Life) order

We can tell that X is lesser than or equal to Y in the NBUE order as represented as, $X \leq_{dmrl} Y$ if

$\frac{l\left(G^{-1}(u)\right)}{m\left(F^{-1}(u)\right)}$ is increasing in u ∈ [0,1].

Note:

It is same as the condition,

$\frac{\frac{1}{EY}\int_{G^{-1}(u)}^{\infty} \bar{G}(x)\, dx}{\frac{1}{EX}\int_{F^{-1}(u)}^{\infty} \bar{F}(x)\, dx}$ is increasing in u ∈ [0,1].

Definition 2.14: NBUE (New Better than Used in Expectation) order

We can tell that X is lesser than or equal to Y in the NBUE order as represented as, $X \leq_{nbue} Y$ if

$$\frac{m\left(F^{-1}(u)\right)}{l\left(G^{-1}(u)\right)} \leq \frac{EX}{EY} \text{ for all u} \in [0,1].$$

It is same as the condition,

$$\frac{1}{EX}\int_{F^{-1}(u)}^{\infty} \bar{F}(x)\, dx \leq \frac{1}{EY}\int_{G^{-1}(u)}^{\infty} \bar{G}(x)\, dx \quad \text{for all u} \in [0,1].$$

Definition 2.15: Fuzzy stochastic order

If $\tilde{X}$ and $\tilde{Y}$ be FRV with fuzzy cumulative distribution functions $\tilde{F}$ and $\tilde{G}$ respectively.

Then, $\tilde{X} \leq_{fst} \tilde{Y}$ if and only if

$$Min\left(\min_{0 \leq \alpha \leq 1} F_\alpha^L(t), \min_{0 \leq \alpha \leq 1} F_\alpha^U(t)\right) \geq Min\left(\min_{0 \leq \alpha \leq 1} G_\alpha^L(t), \min_{0 \leq \alpha \leq 1} G_\alpha^U(t)\right)$$

$$Max\left(\max_{0 \leq \alpha \leq 1} F_\alpha^L(t), \max_{0 \leq \alpha \leq 1} F_\alpha^U(t)\right) \geq Max\left(\max_{\alpha \leq \alpha \leq 1} G_\alpha^L(t), \max_{0 \leq \alpha \leq 1} G_\alpha^U(t)\right)$$

for all t and ∀ $\alpha \in (0,1]$.

Definition 2.16: Fuzzy Convex transform order for uniform FRV

We can tell that $\tilde{X}$ is lesser than or equal to $\tilde{Y}$ in the FCT order represented as, $\tilde{U} \leq_{fct} \tilde{X}$ if,

$$Min\left(\min_{\alpha \leq \beta < 1} G^{-1}F_U(x_\beta^L), \min_{\alpha \leq \beta < 1} G^{-1}F_U(x_\beta^U)\right) \text{ and}$$

$$Max\left(\max_{\alpha \leq \beta < 1} G^{-1}F_U(x_\beta^L), \max_{\alpha \leq \beta < 1} G^{-1}F_U(x_\beta^U)\right), \text{ where } \beta \in [\alpha, 1).$$

is fuzzy convex on support of $\tilde{X}$.

Definition 2.17: Fuzzy convex transform order (FCT)

We can tell that $\tilde{X}$ is lesser than or equal to $\tilde{Y}$ in the FCT order represented as, $\tilde{X} \leq_{fct} \tilde{Y}$ if

$$Min\left(\min_{\alpha \leq \beta < 1} G^{-1}F(x_\beta^L), \min_{\alpha \leq \beta < 1} G^{-1}F(x_\beta^U)\right) \text{ and}$$

$Max\left(\underset{\alpha \le \beta < 1}{max} G^{-1}F(x_\beta^L), \underset{\alpha \le \beta < 1}{max} G^{-1}F(x_\beta^U)\right)$ is fuzzy convex on support of X,

where $\beta \in [\alpha, 1)$.

Definition 2.18: Condition satisfying Fuzzy convex transform order (FCT)

We can tell that $\widetilde{X}$ is lesser than or equal to $\widetilde{Y}$ in the FCT order represented as, $\widetilde{X} \le_{fct} \widetilde{Y}$ if and only if,

$$Min\left(\underset{0<\alpha\le 1}{min} \frac{f\left((F^{-1})_\alpha^L(u)\right)}{g\left((G^{-1})_\alpha^L(u)\right)}, \underset{0<\alpha\le 1}{min} \frac{f\left((F^{-1})_\alpha^U(u)\right)}{g\left((G^{-1})_\alpha^U(u)\right)}\right)$$

$$Max\left(\underset{0<\alpha\le 1}{max} \frac{f\left((F^{-1})_\alpha^L(u)\right)}{g\left((G^{-1})_\alpha^L(u)\right)}, \underset{0<\alpha\le 1}{max} \frac{f\left((F^{-1})_\alpha^U(u)\right)}{g\left((G^{-1})_\alpha^U(u)\right)}\right)$$

is increasing in $u \in [0,1]$.

Where f and g are the fuzzy densities associated with F and G respectively.

Definition 2.19: Fuzzy Star order (F∗)

Suppose that the support of $\widetilde{X}$ is an interval. We can tell that $\widetilde{X}$ is lesser than or equal to $\widetilde{Y}$ in the FS order as represented as, $\widetilde{X} \le_{f*} \widetilde{Y}$ if,

$Min\left(\underset{\alpha\le\beta<1}{min} \frac{G^{-1}F(x_\beta^L)}{(x_\beta^L)}, \underset{\alpha\le\beta<1}{min} \frac{G^{-1}F(x_\beta^U)}{(x_\beta^U)}\right)$ and

$Max\left(\underset{\alpha\le\beta<1}{max} \frac{G^{-1}F(x_\beta^L)}{(x_\beta^L)}, \underset{\alpha\le\beta<1}{max} \frac{G^{-1}F(x_\beta^U)}{(x_\beta^U)}\right)$

increases in $x \ge 0$, $where\ \beta \in [\alpha, 1)$.

Definition 2.20: Fuzzy Super additive order (FSUO)

We can tell that $\widetilde{X}$ is lesser than or equal to $\widetilde{Y}$ in the FSU order as represented as, $\widetilde{X} \le_{su} \widetilde{Y}$ if

Case (i)

$$Min\left(\underset{\alpha\le\beta<1}{min} G^{-1}F\left((x_\beta^L)+(y_\beta^L)\right), \underset{\alpha\le\beta<1}{min} G^{-1}F\left((x_\beta^U)+(y_\beta^U)\right)\right)$$

$$\ge Min\left(\underset{\alpha\le\beta<1}{min} G^{-1}F(x_\beta^L)+G^{-1}F(y_\beta^L), \underset{\alpha\le\beta<1}{min} G^{-1}F(x_\beta^U)+G^{-1}F(y_\beta^U)\right)$$

and

$$Max\left(\underset{\alpha\le\beta<1}{max} G^{-1}F\left((x_\beta^L)+(y_\beta^L)\right), \underset{\alpha\le\beta<1}{max} G^{-1}F\left((x_\beta^U)+(y_\beta^U)\right)\right)$$

$$\ge Max\left(\underset{\alpha\le\beta<1}{max} G^{-1}F(x_\beta^L)+G^{-1}F(y_\beta^L), \underset{\alpha\le\beta<1}{max} G^{-1}F(x_\beta^U)+G^{-1}F(y_\beta^U)\right)$$

Case (ii)

$$Min\left(\underset{\alpha\le\beta<1}{min} G^{-1}F\left((x_\beta^L)+(y_\beta^L)\right), \underset{\alpha\le\beta<1}{min} G^{-1}F\left((x_\beta^U)+(y_\beta^U)\right)\right)$$

$$\ge Min\left(\underset{\alpha\le\beta<1}{min} G^{-1}F(x_\beta^L)+G^{-1}F(y_\beta^L), \underset{\alpha\le\beta<1}{min} G^{-1}F(x_\beta^U)+G^{-1}F(y_\beta^U)\right)$$

and

$$Max\left(\underset{\alpha\le\beta<1}{max} G^{-1}F\left((x_\beta^L)+(y_\beta^L)\right), \underset{\alpha\le\beta<1}{max} G^{-1}F\left((x_\beta^U)+(y_\beta^U)\right)\right)$$

$$\ge Max\left(\underset{\alpha\le\beta<1}{max} G^{-1}F(x_\beta^L)+G^{-1}F(y_\beta^L), \underset{\alpha\le\beta<1}{max} G^{-1}F(x_\beta^U)+G^{-1}F(y_\beta^U)\right)$$

Case (iii)

$$Min\left(\underset{\alpha\le\beta<1}{min} G^{-1}F\left((x_\beta^U)+(y_\beta^U)\right), \underset{\alpha\le\beta<1}{min} G^{-1}F\left((x_\beta^U)+(y_\beta^U)\right)\right)$$

$$\ge Min\left(\underset{\alpha\le\beta<1}{min} G^{-1}F(x_\beta^U)+G^{-1}F(y_\beta^U), \underset{\alpha\le\beta<1}{min} G^{-1}F(x_\beta^U)+G^{-1}F(y_\beta^U)\right)$$

and

$$Max\left(\underset{\alpha\le\beta<1}{max} G^{-1}F\left((x_\beta^U)+(y_\beta^U)\right), \underset{\alpha\le\beta<1}{max} G^{-1}F\left((x_\beta^U)+(y_\beta^U)\right)\right)$$

$$\ge Max\left(\underset{\alpha\le\beta<1}{max} G^{-1}F(x_\beta^U)+G^{-1}F(y_\beta^U), \underset{\alpha\le\beta<1}{max} G^{-1}F(x_\beta^U)+G^{-1}F(y_\beta^U)\right)$$

for all $x \ge 0$ and $y \ge 0$, where $\beta \in (0,1]$.

Definition 2.21: Fuzzy TTT (Total Time on Test) transform

The fuzzy Total time on test (TTT) transform connected with $\widetilde{F}$ is defined by

$$\widetilde{H_{F^{-1}}}_\alpha(u) =$$

$$Min\left[\underset{\alpha\le\beta<1}{min}\int_0^{F^{-1}(u)}\bar{F}(x_\beta^L)\,dx, \underset{\alpha\le\beta<1}{min}\int_0^{F^{-1}(u)}\bar{F}(x_\beta^U)\,dx\right]\text{and}$$

$$Max\left[\underset{\alpha\le\beta<1}{max}\int_0^{F^{-1}(u)}\bar{F}(x_\beta^L)\,dx, \underset{\alpha\le\beta<1}{max}\int_0^{F^{-1}(u)}\bar{F}(x_\beta^U)\,dx\right]$$

where $0 \le u \le 1, \beta \in [\alpha, 1]$.

where $\bar{F} = 1 - F$, the fuzzy survival function associated with $\widetilde{F}$.

The fuzzy Total time on test (TTT) transform associated with G is defined y

$$\widetilde{H_{G^{-1}}}_\beta(u) = Min\left[\underset{\alpha\le\beta<1}{min}\int_0^{G^{-1}(u)}\bar{G}(x_\beta^L)\,dx, \underset{\alpha\le\beta<1}{min}\int_0^{G^{-1}(u)}\bar{G}(x_\beta^U)\,dx\right]$$

and

$$Max\left[\underset{\alpha\le\beta<1}{max}\int_0^{G^{-1}(u)}\bar{G}(x_\beta^L)\,dx, \underset{\alpha\le\beta<1}{max}\int_0^{G^{-1}(u)}\bar{G}(x_\beta^U)\,dx\right]$$

For all $0 \le u \le 1, \beta \in [\alpha, 1)$.

where $\bar{G} = 1 - G$, the fuzzy survival function connected with $\tilde{G}$, where $\beta \in (0,1]$.

Definition 2.22: F-TTT (Total Time on Test) Random variable

Let $\tilde{X}_{ttt}$ and Y_{ttt} be the fuzzy TTT RV, having fuzzy distribution functions $\widetilde{H_F}$ *and* $\widetilde{H_G}$. And $\widetilde{h_F}$ and $\widetilde{h_G}$ be the fuzzy density functions respectively. The inverse of $\widetilde{H_F}$ *and* $\widetilde{H_G}$ of TTT transforms are also fuzzy distribution functions.

The density function $\widetilde{h_F}$ and $\widetilde{h_G}$ of the inverse of the F-TTT transform $\widetilde{H_F}$ *and* $\widetilde{H_G}$ that is connected with F satisfies

$$\widetilde{h_F}_\beta\left(\widetilde{(H_F^{-1})}_\alpha(u)\right)=Min\left(\min_{\alpha\le\beta<1}\frac{f_\beta^L\left((F_\alpha^L)^{-1}(u)\right)}{1-u},\min_{\alpha\le\beta<1}\frac{f_\beta^U\left((F_\alpha^U)^{-1}(u)\right)}{1-u}\right)$$
$$Max\left(\max_{\alpha\le\beta<1}\frac{f_\beta^L\left((F_\alpha^L)^{-1}(u)\right)}{1-u},\max_{\alpha\le\beta<1}\frac{f_\beta^U\left((F_\alpha^U)^{-1}(u)\right)}{1-u}\right)$$

$$\widetilde{h_G}_\beta\left(\widetilde{(H_G^{-1})}_\alpha(u)\right)=Min\left(\min_{\alpha\le\beta<1}\frac{g_\beta^L\left((G_\alpha^L)^{-1}(u)\right)}{1-u},\min_{\alpha\le\beta<1}\frac{g_\beta^U\left((G_\alpha^U)^{-1}(u)\right)}{1-u}\right)$$
$$Max\left(\max_{\alpha\le\beta<1}\frac{g_\beta^L\left((G_\alpha^L)^{-1}(u)\right)}{1-u},\max_{\alpha\le\beta<1}\frac{g_\beta^U\left((G_\alpha^U)^{-1}(u)\right)}{1-u}\right),\text{where}\quad\alpha,\beta\in(0,1].$$

Definition 2.23 Fuzzy DMRL (Decreasing Mean Residual Life) order

We can tell that $\tilde{X}$ is lesser than or equal to $\tilde{Y}$ in the F-DMRL order as represented as, $\tilde{X}\le_{f-dmrl}\tilde{Y}$, if

$$\text{Min}\left(\min_{\alpha\le\beta\le1}\frac{\frac{1}{E(Y_\alpha^L)}\int_{G^{-1}(u)}^{\infty}\bar{G}(x_\beta^L)\,dx}{\frac{1}{E(X_\alpha^L)}\int_{F^{-1}(u)}^{\infty}\bar{F}(x_\beta^L)\,dx},\ \min_{\alpha\le\beta\le1}\frac{\frac{1}{E(Y_\alpha^U)}\int_{G^{-1}(u)}^{\infty}\bar{G}(x_\beta^U)\,dx}{\frac{1}{E(X_\alpha^U)}\int_{F^{-1}(u)}^{\infty}\bar{F}(x_\beta^U)\,dx}\right)\text{ and}$$
$$\text{Max}\left(\max_{\alpha\le\beta\le1}\frac{\frac{1}{E(Y_\alpha^L)}\int_{G^{-1}(u)}^{\infty}\bar{G}(x_\beta^L)\,dx}{\frac{1}{E(X_\alpha^L)}\int_{F^{-1}(u)}^{\infty}\bar{F}(x_\beta^L)\,dx},\ \max_{\alpha\le\beta\le1}\frac{\frac{1}{E(Y_\alpha^U)}\int_{G^{-1}(u)}^{\infty}\bar{G}(x_\beta^U)\,dx}{\frac{1}{E(X_\alpha^U)}\int_{F^{-1}(u)}^{\infty}\bar{F}(x_\beta^U)\,dx}\right)$$

is increasing in u $\in [0,1]$, where $\alpha,\beta\in(0,1]$

Definition 2.24: Fuzzy New Better than Used in Expectation (FNBUE) order

We can tell that $\tilde{X}$ is lesser than or equal to $\tilde{Y}$ in the F-NBUE order as represented as, $\tilde{X}\le_{f-nbue}\tilde{Y}$, if

$$Min\left[\min_{\alpha\le\beta\le1}\frac{1}{E(X_\alpha^L)}\int_{F^{-1}(u)}^{\infty}\bar{F}(x_\beta^L)\,dx,\ \min_{\alpha\le\beta\le1}\frac{1}{E(X_\alpha^U)}\int_{F^{-1}(u)}^{\infty}\bar{F}(x_\beta^U)\,dx\right]$$
$$\le$$
$$Min\left[\min_{\alpha\le\beta\le1}\frac{1}{E(Y_\alpha^L)}\int_{G^{-1}(u)}^{\infty}\bar{G}(x_\beta^L)\,dx,\ \min_{\alpha\le\beta\le1}\frac{1}{E(Y_\alpha^U)}\int_{G^{-1}(u)}^{\infty}\bar{G}(x_\beta^U)\,dx\right]\text{and}$$
$$Max\left[\max_{\alpha\le\beta\le1}\frac{1}{E(X_\alpha^L)}\int_{F^{-1}(u)}^{\infty}\bar{F}(x_\beta^L)\,dx,\ \max_{\alpha\le\beta\le1}\frac{1}{E(X_\alpha^U)}\int_{F^{-1}(u)}^{\infty}\bar{F}(x_\beta^U)\,dx\right]$$
$$\le Max\left[\max_{\alpha\le\beta\le1}\frac{1}{E(Y_\alpha^L)}\int_{G^{-1}(u)}^{\infty}\bar{G}(x_\beta^L)\,dx,\ \max_{\alpha\le\beta\le1}\frac{1}{E(Y_\alpha^U)}\int_{G^{-1}(u)}^{\infty}\bar{G}(x_\beta^U)\,dx\right]$$

for all u $\in$ $[0,1]$ *and* $\alpha,\beta\in(0,1]$.

3. Properties of Fuzzy convex transform order with some related orders:

Theorem. 3.1:

Let $\tilde{X}$ and $\tilde{Y}$ be two FRVs with fuzzy distribution functions $\tilde{F}$ and $\tilde{G}$ respectively. Suppose that the support of X is an interval, we have

(a) If $\tilde{X}\le_{fct}\tilde{Y}$, then $\tilde{X}\le_{f*}\tilde{Y}$

(b) *If* $\tilde{X}\le_{f*}\tilde{Y}$, *then* $\tilde{X}\le_{fsu}\tilde{Y}$

Proof:

Now let us assume $\tilde{X}\le_{fct}\tilde{Y}$.

Then by the definition of FCTO we have,

$$\dot{\phi}(\widetilde{x_\beta})=Min\left(\min_{\alpha\le\beta<1}G^{-1}F(x_\beta^L),\ \min_{\alpha\le\beta<1}G^{-1}F(x_\beta^U)\right)\text{ and}$$
$$Max\left(\max_{\alpha\le\beta<1}G^{-1}F(x_\beta^L),\max_{\alpha\le\beta<1}G^{-1}F(x_\beta^U)\right)$$

is fuzzy convex on the support of F.

Every increasing convex function $\dot{\phi}$ on R^+that passes through the origin is starshaped. $(\dot{\phi}(0)=0)$

So, if the fuzzy convex function, $G^{-1}F(x_\beta^L,x_\beta^U)$ increases in $x\ge0$,

In FCTO condition, multiplying by $\frac{1}{x}$, for $x\ge0$,

$$\Rightarrow Min\left(\min_{\alpha\le\beta<1}\frac{G^{-1}F(x_\beta^L)}{(x_\beta^L)},\ \min_{\alpha\le\beta<1}\frac{G^{-1}F(x_\beta^U)}{(x_\beta^U)}\right)\ and$$
$$Max\left(\max_{\alpha\le\beta<1}\frac{G^{-1}F(x_\beta^L)}{(x_\beta^L)},\max_{\alpha\le\beta<1}\frac{G^{-1}F(x_\beta^U)}{(x_\beta^U)}\right)\text{ increases in }x\ge0.$$

Which implies, $G^{-1}F(x_\beta^L,x_\beta^U)$ is fuzzy starshaped in x, where $\beta\in[\alpha,1)$.

Hence by the definition of FSO, $X\le_{f*}Y$.

So, $X\le_{fct}Y\Rightarrow X\le_{f*}Y$

Every non-negative function that vanishes at 0, and which are increasing and convex on R^+, will also be starshaped on R^+.That is a fuzzy convex function that passes through the origin is fuzzy starshaped.

(b) Now let us assume $\widetilde{X} \leq_{f*} \widetilde{Y}$.

Then by the definition of FSO we have, $G^{-1}F(x_\beta^L, x_\beta^U)$ is fuzzy starshaped in x.

That is,

$Min\left(\underset{\alpha \leq \beta < 1}{min} \frac{G^{-1}F(x_\beta^L)}{(x_\beta^L)}, \underset{\alpha \leq \beta < 1}{min} \frac{G^{-1}F(x_\beta^U)}{(x_\beta^U)}\right)$ *and*

$Max\left(\underset{\alpha \leq \beta < 1}{max} \frac{G^{-1}F(x_\beta^L)}{(x_\beta^L)}, \underset{\alpha \leq \beta < 1}{max} \frac{G^{-1}F(x_\beta^U)}{(x_\beta^U)}\right)$ increases in $x \geq 0$, *where* $\beta \in [\alpha, 1)$.

Choosing any positive values x and y in the support S_F of F, with $x < x + y$ and $y < x + y$ and for $0 < x < x + y$ and since is increasing at x and y,

$Min\left(\underset{\alpha \leq \beta < 1}{min} \frac{G^{-1}F(x_\beta^L)}{(x_\beta^L)}, \underset{\alpha \leq \beta < 1}{min} \frac{G^{-1}F(x_\beta^U)}{(x_\beta^U)}\right)$

$\leq Min\left(\underset{\alpha \leq \beta < 1}{min} \frac{G^{-1}F((x_\beta^L)+(y_\beta^L))}{(x_\beta^L)+(y_\beta^L)}, \underset{\alpha \leq \beta < 1}{min} \frac{G^{-1}F((x_\beta^U)+(y_\beta^U))}{(x_\beta^U)+(y_\beta^U)}\right)$ and

$Max\left(\underset{\alpha \leq \beta < 1}{max} \frac{G^{-1}F(x_\beta^L)}{(x_\beta^L)}, \underset{\alpha \leq \beta < 1}{max} \frac{G^{-1}F(x_\beta^U)}{(x_\beta^U)}\right)$

$\leq Max\left(\underset{\alpha \leq \beta < 1}{max} \frac{G^{-1}F((x_\beta^L)+(y_\beta^L))}{(x_\beta^L)+(y_\beta^L)}, \underset{\alpha \leq \beta < 1}{max} \frac{G^{-1}F((x_\beta^U)+(y_\beta^U))}{(x_\beta^U)+(y_\beta^U)}\right)$ for all $x \geq 0$ and $y \geq 0$.

Similarly, for $0 < y < x + y$, $Min\left(\underset{\alpha \leq \beta < 1}{min} \frac{G^{-1}F(y_\beta^L)}{(x_\beta^L)}, \underset{\alpha \leq \beta < 1}{min} \frac{G^{-1}F(y_\beta^U)}{(x_\beta^U)}\right)$

$\leq Min\left(\underset{\alpha \leq \beta < 1}{min} \frac{G^{-1}F((x_\beta^L)+(y_\beta^L))}{(x_\beta^L)+(y_\beta^L)}, \underset{\alpha \leq \beta < 1}{min} \frac{G^{-1}F((x_\beta^U)+(y_\beta^U))}{(x_\beta^U)+(y_\beta^U)}\right)$ and

$Max\left(\underset{\alpha \leq \beta < 1}{max} \frac{G^{-1}F(y_\beta^L)}{(x_\beta^L)}, \underset{\alpha \leq \beta < 1}{max} \frac{G^{-1}F(y_\beta^U)}{(x_\beta^U)}\right)$

$\leq Max\left(\underset{\alpha \leq \beta < 1}{max} \frac{G^{-1}F((x_\beta^L)+(y_\beta^L))}{(x_\beta^L)+(y_\beta^L)}, \underset{\alpha \leq \beta < 1}{max} \frac{G^{-1}F((x_\beta^U)+(y_\beta^U))}{(x_\beta^U)+(y_\beta^U)}\right)$

Now adding both, we have,

(i) $Min\left(\underset{\alpha \leq \beta < 1}{min} G^{-1}F\left((x_\beta^L)+(y_\beta^L)\right), \underset{\alpha \leq \beta < 1}{min} G^{-1}F\left((x_\beta^U)+(y_\beta^U)\right)\right)$

$\geq Min\left(\underset{\alpha \leq \beta < 1}{min} G^{-1}F(x_\beta^L) + G^{-1}F(y_\beta^L), \underset{\alpha \leq \beta < 1}{min} G^{-1}F(x_\beta^U) + G^{-1}F(y_\beta^U)\right)$

and

$Max\left(\underset{\alpha \leq \beta < 1}{max} G^{-1}F\left((x_\beta^L)+(y_\beta^L)\right), \underset{\alpha \leq \beta < 1}{max} G^{-1}F\left((x_\beta^U)+(y_\beta^U)\right)\right)$

$\geq Max\left(\underset{\alpha \leq \beta < 1}{max} G^{-1}F(x_\beta^L) + G^{-1}F(y_\beta^L), \underset{\alpha \leq \beta < 1}{max} G^{-1}F(x_\beta^U) + G^{-1}F(y_\beta^U)\right)$

(ii) $Min\left(\underset{\alpha \leq \beta < 1}{min} G^{-1}F\left((x_\beta^L)+(y_\beta^L)\right), \underset{\alpha \leq \beta < 1}{min} G^{-1}F\left((x_\beta^U)+(y_\beta^U)\right)\right)$

$\geq Min\left(\underset{\alpha \leq \beta < 1}{min} G^{-1}F(x_\beta^L) + G^{-1}F(y_\beta^L), \underset{\alpha \leq \beta < 1}{min} G^{-1}F(x_\beta^U) + G^{-1}F(y_\beta^U)\right)$

and

$Max\left(\underset{\alpha \leq \beta < 1}{max} G^{-1}F\left((x_\beta^L)+(y_\beta^L)\right), \underset{\alpha \leq \beta < 1}{max} G^{-1}F\left((x_\beta^U)+(y_\beta^U)\right)\right)$

$\geq Max\left(\underset{\alpha \leq \beta < 1}{max} G^{-1}F(x_\beta^L) + G^{-1}F(y_\beta^L), \underset{\alpha \leq \beta < 1}{max} G^{-1}F(x_\beta^U) + G^{-1}F(y_\beta^U)\right)$

(iii) $Min\left(\underset{\alpha \leq \beta < 1}{min} G^{-1}F\left((x_\beta^U)+(y_\beta^U)\right), \underset{\alpha \leq \beta < 1}{min} G^{-1}F\left((x_\beta^U)+(y_\beta^U)\right)\right)$

$\geq Min\left(\underset{\alpha \leq \beta < 1}{min} G^{-1}F(x_\beta^U) + G^{-1}F(y_\beta^U), \underset{\alpha \leq \beta < 1}{min} G^{-1}F(x_\beta^U) + G^{-1}F(y_\beta^U)\right)$

and

$Max\left(\underset{\alpha \leq \beta < 1}{max} G^{-1}F\left((x_\beta^U)+(y_\beta^U)\right), \underset{\alpha \leq \beta < 1}{max} G^{-1}F\left((x_\beta^U)+(y_\beta^U)\right)\right)$

$\geq Max\left(\underset{\alpha \leq \beta < 1}{max} G^{-1}F(x_\beta^U) + G^{-1}F(y_\beta^U), \underset{\alpha \leq \beta < 1}{max} G^{-1}F(x_\beta^U) + G^{-1}F(y_\beta^U)\right)$

for all $x \geq 0$ and $y \geq 0$.

Which implies that $G^{-1}F(x_\beta^L, x_\beta^U)$ is fuzzy super additive in x, where $\beta \in (0,1]$.

Hence by the definition of FSU order, $\widetilde{X} \leq_{fsu} \widetilde{Y}$.

So, $\widetilde{X} \leq_{f*} \widetilde{Y} \Rightarrow \widetilde{X} \leq_{fsu} \widetilde{Y}$.

Every non-negative function that vanishes at 0, and which are increasing and starshaped on R^+, will also be superadditive on R^+,

Therefore, $\widetilde{X} \leq_{fct} \widetilde{Y} \Rightarrow \widetilde{X} \leq_{f*} \widetilde{Y} \Rightarrow \widetilde{X} \leq_{fsu} \widetilde{Y}$.

Theorem. 3.2:

Let $\widetilde{X}$ and $\widetilde{Y}$ be two non-negative FRVs with absolutely continuous fuzzy distribution functions $\widetilde{F}$ and $\widetilde{G}$ having 0 as the left endpoint of their supports. Then, $\widetilde{X} \leq_{fct} \widetilde{Y} \Leftrightarrow \widetilde{X}_{ttt} \leq_{fct} Y_{ttt}$

Proof:

Note that, $\widetilde{X} \leq_{fct} \widetilde{Y}$ if and only if,

$Min\left(\underset{\alpha \leq \beta \leq 1}{min} \frac{f_\beta^L\left((F_\alpha^L)^{-1}(u)\right)}{g_\beta^L\left((G_\alpha^L)^{-1}(u)\right)}, \underset{\alpha \leq \beta \leq 1}{min} \frac{f_\beta^U\left((F_\alpha^U)^{-1}(u)\right)}{g_\beta^U\left((G_\alpha^U)^{-1}(u)\right)}\right)$

$$Max\left(\underset{\alpha\le\beta\le 1}{max}\frac{f_\beta^L\left((F_\alpha^L)^{-1}(u)\right)}{g_\beta^L\left((G_\alpha^L)^{-1}(u)\right)},\underset{\alpha\le\beta\le 1}{max}\frac{f_\beta^U\left((F_\alpha^U)^{-1}(u)\right)}{g_\beta^U\left((G_\alpha^U)^{-1}(u)\right)}\right)$$

is increasing in u ∈ [0,1], $\alpha,\beta\in(0,1]$. ----------[3.1]

Where $\tilde{f}$ and $\tilde{g}$ are the fuzzy densities associated with $\widetilde{F}$ and $\tilde{G}$ respectively.

i.e., $Min\left(\underset{\alpha\le\beta<1}{min}G^{-1}F(x_\beta^L),\underset{\alpha\le\beta<1}{min}G^{-1}F(x_\beta^U)\right)$ and

$$Max\left(\underset{\alpha\le\beta<1}{max}G^{-1}F(x_\beta^L),\underset{\alpha\le\beta<1}{max}G^{-1}F(x_\beta^U)\right)$$

$$\Leftrightarrow Min\left(\underset{\alpha\le\beta\le 1}{min}\frac{f_\beta^L\left((F_\alpha^L)^{-1}(u)\right)}{g_\beta^L\left((G_\alpha^L)^{-1}(u)\right)},\underset{\alpha\le\beta\le 1}{min}\frac{f_\beta^U\left((F_\alpha^U)^{-1}(u)\right)}{g_\beta^U\left((G_\alpha^U)^{-1}(u)\right)}\right)$$

$$Max\left(\underset{\alpha\le\beta\le 1}{max}\frac{f_\beta^L\left((F_\alpha^L)^{-1}(u)\right)}{g_\beta^L\left((G_\alpha^L)^{-1}(u)\right)},\underset{\alpha\le\beta\le 1}{max}\frac{f_\beta^U\left((F_\alpha^U)^{-1}(u)\right)}{g_\beta^U\left((G_\alpha^U)^{-1}(u)\right)}\right)$$

is increasing in u ∈ [0,1], $\alpha,\beta\in(0,1]$.

Now from the definition of F-TTT random variable, we know (3) & (4).

Hence from [3.1] and using the definition of F-TTT random variable it is seen that,

$$\Leftrightarrow Min\left(\underset{\alpha\le\beta\le 1}{min}\frac{(h_F)_\beta^L\left((H_F^{-1})_\alpha^L(u)\right)}{(h_G)_\beta^L\left((H_G^{-1})_\alpha^L(u)\right)},\underset{\alpha\le\beta\le 1}{min}\frac{(h_F)_\beta^U\left((H_F^{-1})_\alpha^U(u)\right)}{(h_G)_\beta^U\left((H_G^{-1})_\alpha^U(u)\right)}\right)$$

$$Max\left(\underset{\alpha\le\beta\le 1}{max}\frac{(h_F)_\beta^L\left((H_F^{-1})_\alpha^L(u)\right)}{(h_G)_\beta^L\left((H_G^{-1})_\alpha^L(u)\right)},\underset{\alpha\le\beta\le 1}{max}\frac{(h_F)_\beta^U\left((H_F^{-1})_\alpha^U(u)\right)}{(h_G)_\beta^U\left((H_G^{-1})_\alpha^U(u)\right)}\right),\ \alpha,\beta\in(0,1].$$

is increasing in u ∈ [0,1], Where h_F and h_G are the fuzzy density functions.

Hence from the definition of TTT transform, substituting the values above equation becomes,

$$\Leftrightarrow$$

Min $\left(\underset{\alpha\le\beta<1}{min}\frac{(h_F)_\beta^L\left(\int_0^{F^{-1}(u)}\bar{F}(x_\beta^L)\,dx\right)}{(h_G)_\beta^L\left(\int_0^{G^{-1}(u)}\bar{G}(x_\beta^L)\,dx\right)},\underset{\alpha\le\beta<1}{min}\frac{(h_F)_\beta^U\left(\int_0^{F^{-1}(u)}\bar{F}(x_\beta^U)\,dx\right)}{(h_G)_\beta^U\left(\int_0^{G^{-1}(u)}\bar{G}(x_\beta^U)\,dx\right)}\right)$ and

Max $\left(\underset{\alpha\le\beta<1}{max}\frac{(h_F)_\beta^L\left(\int_0^{F^{-1}(u)}\bar{F}(x_\beta^L)\,dx\right)}{(h_G)_\beta^L\left(\int_0^{G^{-1}(u)}\bar{G}(x_\beta^L)\,dx\right)},\underset{\alpha\le\beta<1}{max}\frac{(h_F)_\beta^U\left(\int_0^{F^{-1}(u)}\bar{F}(x_\beta^U)\,dx\right)}{(h_G)_\beta^U\left(\int_0^{G^{-1}(u)}\bar{G}(x_\beta^U)\,dx\right)}\right)$, $\beta\in[\alpha,1)$.

$$\Rightarrow \widetilde{X}_{ttt}\le_{fct}\widetilde{Y}_{ttt}$$

We know that $\widetilde{X}_{ttt}$ and Y_{ttt} be the fuzzy TTT RV, having fuzzy distribution functions $\widetilde{H_F}$ *and* $\widetilde{H_G}$. And $\widetilde{h_F}$ and $\widetilde{h_G}$ be the fuzzy density functions respectively. The inverse of $\widetilde{H_F}$ *and* $\widetilde{H_G}$ of TTT transforms are also fuzzy distribution functions.

Hence, $\widetilde{X}\le_{fct}\widetilde{Y}\Leftrightarrow\widetilde{X}_{ttt}\le_{fct}\widetilde{Y}_{ttt}$.

Counter example:

Let $\widetilde{X}$ be a non-negative FRVs with an absolutely continuous distribution function. Then $\widetilde{X}$ has a decreasing density if and only if $\widetilde{U}\le_{fct}\widetilde{X}$, where $\widetilde{U}$ is a uniform [0,1] FRV.

Theorem.3.3:

Let $\widetilde{X}$ and Y be two FRVs. Then,

(a) If $\widetilde{X}\le_{fct}\widetilde{Y}$, then $\widetilde{X}\le_{fdmrl}\widetilde{Y}$.

(b) If $\widetilde{X}\le_{f-dmrl}\widetilde{Y}$, then $\widetilde{X}\le_{f-nbue}\widetilde{Y}$.

Proof:

(a) Let $\widetilde{X}$ and $\widetilde{Y}$ be two FRVs, each with the support of the form [0, a).

The equilibrium fuzzy survival functions associated with $\widetilde{F}$ and $\tilde{G}$ as is defined by,

$$\bar{F}_e(\widetilde{x_\beta})=Min\left(\underset{0\le\alpha\le 1}{min}\int_x^\infty\frac{\bar{F}(t)}{E(X_\alpha^L)}dt,\underset{0\le\alpha\le 1}{min}\int_x^\infty\frac{\bar{F}(t)}{E(X_\alpha^U)}dt\right)\text{ and}$$

$$Max\left(\underset{0\le\alpha\le 1}{max}\int_x^\infty\frac{\bar{F}(t)}{E(X_\alpha^L)}dt,\underset{0\le\alpha\le 1}{max}\int_x^\infty\frac{\bar{F}(t)}{E(X_\alpha^U)}dt\right)$$

$$\bar{G}_e(\widetilde{x_\beta})=Min\left(\underset{0\le\alpha\le 1}{min}\int_x^\infty\frac{\bar{G}(t)}{E(Y_\alpha^L)}dt,\underset{0\le\alpha\le 1}{min}\int_x^\infty\frac{\bar{G}(t)}{E(Y_\alpha^U)}dt\right)\text{ and}$$

$$Max\left(\underset{0\le\alpha\le 1}{max}\int_x^\infty\frac{\bar{G}(t)}{E(Y_\alpha^L)}dt,\underset{0\le\alpha\le 1}{max}\int_x^\infty\frac{\bar{G}(t)}{E(Y_\alpha^U)}dt\right),\alpha\in(0,1].$$

Now assume $\widetilde{X}\le_{fct}\widetilde{Y}$,

Hence, $\delta(\widetilde{x_\beta})=Min\left(\underset{\alpha\le\beta<1}{min}G^{-1}F(x_\beta^L),\underset{\alpha\le\beta<1}{min}G^{-1}F(x_\beta^U)\right)$ and

$$Max\left(\underset{\alpha\le\beta<1}{max}G^{-1}F(x_\beta^L),\underset{\alpha\le\beta<1}{max}G^{-1}F(x_\beta^U)\right)$$

$$=Min\left(\underset{\alpha\le\beta<1}{min}\bar{G}^{-1}\left(\bar{F}(x_\beta^L)\right),\underset{\alpha\le\beta<1}{min}\bar{G}^{-1}\left(\bar{F}(x_\beta^U)\right)\right)$$

and

$$Max\left(\underset{\alpha\le\beta<1}{max}\bar{G}^{-1}\left(\bar{F}(x_\beta^L)\right),\underset{\alpha\le\beta<1}{max}G^{-1}\left(\bar{F}(x_\beta^U)\right)\right)$$

And let,

$\widetilde{\gamma_\beta}(u) \equiv$

$$Min\left(\min_{0\le\alpha\le 1}(\bar{F}_e)_\alpha^L\left(\delta^{-l}\left(\left(\bar{G}_e{}^{-l}\right)_\alpha^L(u)\right)\right),\min_{0\le\alpha\le 1}(\bar{F}_e)_\alpha^U\left(\delta^{-l}\left(\left(\bar{G}_e{}^{-l}\right)_\alpha^U(u)\right)\right)\right)$$

$$Max\left(\max_{0\le\alpha\le 1}(\bar{F}_e)_\alpha^L\left(\delta^{-l}\left(\left(\bar{G}_e{}^{-l}\right)_\alpha^L(u)\right)\right),\max_{0\le\alpha\le 1}(\bar{F}_e)_\alpha^U\left(\delta^{-l}\left(\left(\bar{G}_e{}^{-l}\right)_\alpha^U(u)\right)\right)\right)$$

for all $u\in[0,1]$ *where* $\alpha\in(0,1]$.

Suppose if δ and γ are differentiable, then

$\widetilde{\gamma_\beta'}(u)$

$$\equiv Min\left(\min_{\alpha\le\beta\le 1}\frac{E(X_\alpha^L)}{E(Y_\alpha^L)}\left[\frac{d}{dx}\,\delta^{-l}(x_\beta^L)\right]_{x=\bar{G}_e{}^{-l}(u)},\min_{\alpha\le\beta\le 1}\frac{E(X_\alpha^U)}{E(Y_\alpha^U)}\left[\frac{d}{dx}\,\delta^{-l}(x_\beta^U)\right]_{x=\bar{G}_e{}^{-l}(u)}\right)$$

$$Max\left(\max_{\alpha\le\beta\le 1}\frac{E(X_\alpha^L)}{E(Y_\alpha^L)}\left[\frac{d}{dx}\,\delta^{-l}(x_\beta^L)\right]_{x=\bar{G}_e{}^{-l}(u)},\max_{\alpha\le\beta\le 1}\frac{E(X_\alpha^U)}{E(Y_\alpha^U)}\left[\frac{d}{dx}\,\delta^{-l}(x_\beta^U)\right]_{x=\bar{G}_e{}^{-l}(u)}\right)$$

----------[3.2]

We assume that $\widetilde{\gamma_\beta}$ is convex.

Then from [3.2], $\widetilde{\gamma_\beta}$ is fuzzy convex, and hence $\widetilde{\gamma_\beta}$ is fuzzy starshaped.

That is,

$$Min\left(\min_{0\le\alpha\le 1}\frac{(\bar{F}_e)_\alpha^L\left(\left(F^{-l}\right)_\alpha^L\left(G_\alpha^L\left(\left(G_e{}^{-l}\right)_\alpha^L(u)\right)\right)\right)}{u},\min_{0\le\alpha\le 1}\frac{(\bar{F}_e)_\alpha^U\left(\left(F^{-l}\right)_\alpha^U\left(G_\alpha^U\left(\left(G_e{}^{-l}\right)_\alpha^U(u)\right)\right)\right)}{u}\right)$$

$$Max\left(\max_{0\le\alpha\le 1}\frac{(\bar{F}_e)_\alpha^L\left(\left(F^{-l}\right)_\alpha^L\left(G_\alpha^L\left(\left(G_e{}^{-l}\right)_\alpha^L(u)\right)\right)\right)}{u},\max_{0\le\alpha\le 1}\frac{(\bar{F}_e)_\alpha^U\left(\left(F^{-l}\right)_\alpha^U\left(G_\alpha^U\left(\left(G_e{}^{-l}\right)_\alpha^U(u)\right)\right)\right)}{u}\right)$$

is increasing in $u\in[0,1]$, *where* $\alpha\in(0,1]$.

Similarly, $Min\left(\min_{0<\alpha\le 1}\frac{(\bar{F}_e)_\alpha^L\left(\left(F^{-l}\right)_\alpha^L(u)\right)}{(\bar{G}_e)_\alpha^L\left(\left(G^{-l}\right)_\alpha^L(u)\right)},\min_{0<\alpha\le 1}\frac{(\bar{F}_e)_\alpha^U\left(\left(F^{-l}\right)_\alpha^U(u)\right)}{(\bar{G}_e)_\alpha^U\left(\left(G^{-l}\right)_\alpha^U(u)\right)}\right)$

$$Max\left(\max_{0<\alpha\le 1}\frac{(\bar{F}_e)_\alpha^L\left(\left(F^{-l}\right)_\alpha^L(u)\right)}{(\bar{G}_e)_\alpha^L\left(\left(G^{-l}\right)_\alpha^L(u)\right)},\max_{0<\alpha\le 1}\frac{(\bar{F}_e)_\alpha^U\left(\left(F^{-l}\right)_\alpha^U(u)\right)}{(\bar{G}_e)_\alpha^U\left(\left(G^{-l}\right)_\alpha^U(u)\right)}\right)$$

is decreasing in $u\in[0,1]$. *where* $\alpha\in(0,1]$.

Which implies

$$\text{Min}\left(\min_{\alpha\le\beta\le 1}\frac{\frac{1}{E(Y_\alpha^L)}\int_{G^{-1}(u)}^{\infty}\bar{G}(x_\beta^L)\,dx}{\frac{1}{E(X_\alpha^L)}\int_{F^{-1}(u)}^{\infty}\bar{F}(x_\beta^L)\,dx},\ \min_{\alpha\le\beta\le 1}\frac{\frac{1}{E(Y_\alpha^U)}\int_{G^{-1}(u)}^{\infty}\bar{G}(x_\beta^U)\,dx}{\frac{1}{E(X_\alpha^U)}\int_{F^{-1}(u)}^{\infty}\bar{F}(x_\beta^U)\,dx}\right)\ \text{and}$$

$$\text{Max}\left(\max_{\alpha\le\beta\le 1}\frac{\frac{1}{E(Y_\alpha^L)}\int_{G^{-1}(u)}^{\infty}\bar{G}(x_\beta^L)\,dx}{\frac{1}{E(X_\alpha^L)}\int_{F^{-1}(u)}^{\infty}\bar{F}(x_\beta^L)\,dx},\ \max_{\alpha\le\beta\le 1}\frac{\frac{1}{E(Y_\alpha^U)}\int_{G^{-1}(u)}^{\infty}\bar{G}(x_\beta^U)\,dx}{\frac{1}{E(X_\alpha^U)}\int_{F^{-1}(u)}^{\infty}\bar{F}(x_\beta^U)\,dx}\right)$$

is increasing in $u\in[0,1]$,where $\alpha,\beta\in(0,1]$.

$\Rightarrow \widetilde{X}\le_{f-dmrl}\widetilde{Y}$.

(b) Assume, $\widetilde{X}\le_{f-dmrl}\widetilde{Y}$

Then by the definition of F-DMRL,

$$\text{Min}\left(\min_{\alpha\le\beta<1}\frac{\frac{1}{E(Y_\alpha^L)}\int_{G^{-1}(u)}^{\infty}\bar{G}(x_\beta^L)\,dx}{\frac{1}{E(X_\alpha^L)}\int_{F^{-1}(u)}^{\infty}\bar{F}(x_\beta^L)\,dx},\ \min_{\alpha\le\beta<1}\frac{\frac{1}{E(Y_\alpha^U)}\int_{G^{-1}(u)}^{\infty}\bar{G}(x_\beta^U)\,dx}{\frac{1}{E(X_\alpha^U)}\int_{F^{-1}(u)}^{\infty}\bar{F}(x_\beta^U)\,dx}\right)\ \text{and}$$

$$\text{Max}\left(\max_{\alpha\le\beta<1}\frac{\frac{1}{E(Y_\alpha^L)}\int_{G^{-1}(u)}^{\infty}\bar{G}(x_\beta^L)\,dx}{\frac{1}{E(X_\alpha^L)}\int_{F^{-1}(u)}^{\infty}\bar{F}(x_\beta^L)\,dx},\ \max_{\alpha\le\beta<1}\frac{\frac{1}{E(Y_\alpha^U)}\int_{G^{-1}(u)}^{\infty}\bar{G}(x_\beta^U)\,dx}{\frac{1}{E(X_\alpha^U)}\int_{F^{-1}(u)}^{\infty}\bar{F}(x_\beta^U)\,dx}\right)$$

is increasing in u $\in[0,1]$, where $\alpha,\beta\in(0,1]$.

Which implies that,

$$Min\left[\min_{\alpha\le\beta<1}\frac{1}{E(X_\alpha^L)}\int_{F^{-1}(u)}^{\infty}\bar{F}(x_\beta^L)\,dx,\min_{\alpha\le\beta<1}\frac{1}{E(X_\alpha^U)}\int_{F^{-1}(u)}^{\infty}\bar{F}(x_\beta^U)\,dx\right]$$

$\le$

$$Min\left[\min_{\alpha\le\beta<1}\frac{1}{E(Y_\alpha^L)}\int_{G^{-1}(u)}^{\infty}\bar{G}(x_\beta^L)\,dx,\min_{\alpha\le\beta<1}\frac{1}{E(Y_\alpha^U)}\int_{G^{-1}(u)}^{\infty}\bar{G}(x_\beta^U)\,dx\right]\text{and}$$

$$Max\left[\max_{\alpha\le\beta<1}\frac{1}{E(X_\alpha^L)}\int_{F^{-1}(u)}^{\infty}\bar{F}(x_\beta^L)\,dx,\max_{\alpha\le\beta<1}\frac{1}{E(X_\alpha^U)}\int_{F^{-1}(u)}^{\infty}\bar{F}(x_\beta^U)\,dx\right]$$

$$\le Max\left[\max_{\alpha\le\beta<1}\frac{1}{E(Y_\alpha^L)}\int_{G^{-1}(u)}^{\infty}\bar{G}(x_\beta^L)\,dx,\max_{\alpha\le\beta<1}\frac{1}{E(Y_\alpha^U)}\int_{G^{-1}(u)}^{\infty}\bar{G}(x_\beta^U)\,dx\right]$$

for all u $\in$ [0,1] *and* $\alpha,\beta\in(0,1]$.

Hence by the definition of F-NBUE order, the above inequality satisfies the condition.

$\Rightarrow \widetilde{X}\le_{f-nbue}\widetilde{Y}$

Therefore, $\widetilde{X} \leq_{fct} \widetilde{Y} \Rightarrow \widetilde{X} \leq_{f-dmrl} \widetilde{Y} \Rightarrow \widetilde{X} \leq_{f-nbue} \widetilde{Y}$.

4. Applications of Fuzzy convex Transform Order:

In this section, we have discussed an application related to fuzzy convex transform order in Poisson process through an example. Using FRVS enables to understand more uncertainness in the process. Poisson process is a widely used stochastic process, which is a kind of counting process. Here we considered two non-homogeneous Poisson process, $M_1 = \{M_1(t), t \geq 0\}$ and $M_2 = \{M_2(t), t \geq 0\}$ with mean functions $\Phi_1 = -log\ \bar{F}$ and $\Phi_2 = -log\ \bar{G}$ respectively. This example relates to the ordering of the FRVs $\widetilde{X}$ and $\widetilde{Y}$ to the ordering of their successive epoch times. It turns out that the FCT ordering of the first two epoch times implies the FCT ordering of all the corresponding later epoch times.

Example 4.1:

Let $\widetilde{X}$ and $\widetilde{Y}$ be two absolutely continuous non-negative FRVs with fuzzy survival functions, $\bar{F}$ and $\bar{G}$ respectively. Denote $\widetilde{\Phi_1} = -log\ \bar{F}$ and $\widetilde{\Phi_2} = -log\ \bar{G}$, j=1,2. Consider two non-homogeneous Poisson process, $M_1 = \{M_1(t), t \geq 0\}$ and $M_2 = \{M_2(t), t \geq 0\}$ with fuzzy mean functions $\widetilde{\Phi_1}$ and $\widetilde{\Phi_2}$ respectively. Let $T_{j,1}$, $T_{j,2}$,…be successive epoch times of process $M_j, j = 1,2$. Note that $X =_{fst} T_{1,1}$ and $X =_{fst} T_{2,1}$.

It turns out that the FCT ordering of the first two epoch times implies the FCT ordering of all the corresponding later epoch times.

That is, if $\widetilde{X} \leq_{fct} \widetilde{Y}$ then $T_{1,n} \leq_{fct} T_{2,n}$, n≥ *1* and $\forall\ \alpha \in (0,1]$.

We know that $\{M(t), t \geq 0\}$ is a non-homogeneous Poisson process with fuzzy mean function $\widetilde{\Phi}$ (that is $\widetilde{\Phi} = E[M(t)]$, let $T_1, T_2, \ldots$ be the successive epoch times.

To check this, we first fix for n≥ *1*.

Let us denote $\tilde{F}_{1,n}$ and $\tilde{F}_{2,n}$ as the fuzzy distribution functions of $T_{1,n}$ and $T_{2,n}$ respectively.

The survival $\bar{F}_{1,n}$ of $T_{1,n}$ be,

$\bar{F}_{1,n}(t) = P(T_{1,n} > t) = \sum_{i=0}^{n-1} \frac{(\Phi_1(t))^i}{i!} e^{-\Phi_1(t)} = \bar{\Gamma}_n \widetilde{\Phi_1}(t),\ t \geq 0.$

Then the fuzzy distribution function be

$\tilde{F}_{1,n}(t) = \psi_n \tilde{F}(t)$ and

$\tilde{F}_{2,n}(t) = \psi_n \tilde{G}(t)$

where $\psi_n(v) \equiv \Gamma_n(-log(1-v)), v \in [0,1]$.

Therefore,

$\tilde{F}_{2,n}^{-1}(\tilde{F}_{1,n}(t))$

$= Min\left(\min_{0 \leq \alpha \leq 1} (\psi_n(G_\alpha^L))^{-1}(\psi_n F_\alpha^L(t)),\ \min_{0 \leq \alpha \leq 1} (\psi_n(G_\alpha^U))^{-1}(\psi_n F_\alpha^U(t))\right)$

$Max\left(\max_{0 \leq \alpha \leq 1} (\psi_n(G_\alpha^L))^{-1}(\psi_n F_\alpha^L(t)),\ \max_{0 \leq \alpha \leq 1} (\psi_n(G_\alpha^U))^{-1}(\psi_n F_\alpha^U(t))\right)$

$= Min\left(\min_{0 \leq \alpha \leq 1} (G_\alpha^L)^{-1} F_\alpha^L(t),\ \min_{0 \leq \alpha \leq 1} (G_\alpha^U)^{-1} F_\alpha^U(t)\right)$

$Max\left(\max_{0 \leq \alpha \leq 1} (G_\alpha^L)^{-1} F_\alpha^L(t),\ \max_{0 \leq \alpha \leq 1} (G_\alpha^U)^{-1} F_\alpha^U(t)\right)$ is fuzzy convex, where $t \geq 0$.

Hence, $\widetilde{X} \leq_{fct} \widetilde{Y}$ then $T_{1,n} \leq_{fct} T_{2,n}$, n≥ *1* and $\forall\ \alpha \in (0,1]$.

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