Regular Spaces via n-Cylindrical Neutrosophic Nano $\delta\beta$-open Sets with an Application to Nanocarrier Selection in Drug Delivery

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Abstract: In this paper, we introduce n-Cylindrical neutrosophic nano regular spaces and their δ, δ pre, δ semi, $\delta\alpha$ and $\delta\beta$ variants. Since an n-Cylindrical neutrosophic set and its complement may intersect, these spaces are defined by means of n-Cylindrical neutrosophic points and separated sets. We characterize $\delta\beta$-regular spaces through $\delta\beta$-closures and $\delta\beta$-neighbourhoods, show that every $\delta\beta$-open set in such a space is a union of $\delta\beta$-closed sets, and prove that this regularity is preserved by suitable mappings. We also introduce strongly and strongly * regular spaces, establish their characterizations and the relations among the classes, give examples showing that some implications are not reversible, and show that strong $\delta\beta$-regularity is a very restrictive condition. As a real life application, we consider the selection of nanocarriers in drug delivery. The risk of each formulation is evaluated by experts as an n-Cylindrical neutrosophic number, whose evidence for and against the risk are dependent components and whose indeterminacy is independent. Using the lower and upper approximations and the boundary, we propose an algorithm that identifies the core attributes by comparing n-Cylindrical neutrosophic nano topologies and ranks the formulations by a score function. In an example with six formulations, the release profile and the biocompatibility of the carrier are the core attributes, the decision is stable under small changes of the data, and the resulting nano topology is $\delta\beta$-regular but not regular.

Keywords: n-Cylindrical neutrosophic nano $\delta\beta$-open set, n-Cylindrical neutrosophic point, separated sets, n-Cylindrical neutrosophic nano $\delta\beta$-regular space, strongly n-Cylindrical neutrosophic nano $\delta\beta$-regular space, core attributes, drug delivery, nanocarrier selection.

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1 Introduction

Zadeh [25] introduced fuzzy sets in 1965 by assigning to each element a membership degree in $[0,1]$. Atanassov [1, 2] extended this idea to intuitionistic fuzzy sets, in which each element has a membership degree and a non-membership degree whose sum does not exceed (i) Chang [4] introduced fuzzy topological spaces, and Çoker [5] extended topological notions to intuitionistic fuzzy sets.

The condition on the two degrees was relaxed by Yager [22, 23, 24], who introduced Pythagorean fuzzy sets, where the sum of the squares of the membership and non-membership degrees does not exceed 1; Olgun et al. [10] studied the corresponding topological spaces. Senapati and Yager [17] further relaxed it to the sum of the cubes, obtaining Fermatean fuzzy sets. Smarandache [18, 19] introduced neutrosophic sets, in which truth, indeterminacy and falsity degrees are assigned independently, so that incomplete and inconsistent information can be handled. Neutrosophic topological spaces were introduced by Salama and Alblowi [14], and fuzzy neutrosophic topological spaces were studied in [3, 21]. Combining these ideas, Saranya Kumari et al. [15, 16] introduced n-Cylindrical fuzzy neutrosophic sets and topological spaces, in which the positive and negative membership degrees satisfy $\alpha^n + \gamma^n \leq 1$ for a fixed n, while the indeterminacy degree β is independent. For $n = 2$ and $n = 3$, this condition is the Pythagorean and Fermatean condition, respectively.

Pawlak [13] introduced rough sets, in which a set is described by its lower and upper approximations with respect to an equivalence relation. Based on these approximations, Lellis Thivagar and Richard [7] introduced nano topology, in which the open sets are generated by the lower and upper approximations and the boundary of a set, so that a nano topology has at most five open sets. Lellis Thivagar et al. [8] introduced neutrosophic nano topology. Pankajam and Kavitha [12] studied δ-open sets in nano topological spaces, Padma et al. [11] studied nano M-open sets, and further neutrosophic nano open sets were investigated in [20]. Mahalakshmi et al. [9] introduced n-Cylindrical neutrosophic nano topological spaces, and Kayalvizhi and Alagesan [6] studied M-open sets in these spaces.

Regularity is one of the basic separation properties of a topological space, and in general topology

it has been studied with many classes of generalized open sets. In an n-Cylindrical neutrosophic setting, however, the classical definition cannot be transferred directly: the intersection of an n-Cylindrical neutrosophic set and its complement need not be the empty set, and "$x \notin A$" carries little information, since a set and its complement may both contain a point. Hence suitable notions of points and of separated sets are needed.

Nano topology is also a practical tool, since its lower and upper approximations describe what is certainly and what is possibly true of the objects that cannot be distinguished by the available attributes, and the attributes that change the nano topology are the decisive ones [7, 8]. A field in which such a tool is useful is drug delivery technology, where a drug is loaded into a nanocarrier, such as a liposome, a polymeric nanoparticle, a solid lipid nanoparticle or a dendrimer, so that it reaches the target tissue in the required amount. The suitability of a formulation depends on attributes such as particle size, surface charge, release profile and biocompatibility, and its preclinical evaluation is uncertain: evidence for and against a therapeutic risk is often strong at the same time, so that the two degrees may violate the intuitionistic and even the Pythagorean condition, while inconclusive test results form an independent source of indeterminacy. These are exactly the features of n-Cylindrical neutrosophic sets.

Research Gap: To the best of our knowledge, regular spaces defined by δ-type open sets, such as n-Cylindrical neutrosophic nano $\delta\beta$-regular spaces and strongly n-Cylindrical neutrosophic nano $\delta\beta$-regular spaces, have not yet been studied in n-Cylindrical neutrosophic nano topological spaces, and n-Cylindrical neutrosophic nano topology has not yet been applied to the identification of decisive attributes in drug delivery.

In this paper, we introduce n-Cylindrical neutrosophic nano regular spaces and their δ, δ pre, δ semi, $\delta\alpha$ and $\delta\beta$ variants, together with their strongly and strongly * versions, in n-Cylindrical neutrosophic nano topological spaces ($nCy\mathcal{N}\mathfrak{N}ts$'s), and we apply n-Cylindrical neutrosophic nano topology to the selection of nanocarriers in drug delivery. The main contributions are as follows.

(i) We define n-Cylindrical neutrosophic points, separated sets and $nCy\mathcal{N}\mathfrak{N}$ (resp. δ, δ pre, δ semi, $\delta\alpha$ and $\delta\beta$) regular spaces, characterize $nCy\mathcal{N}\mathfrak{N}\delta\beta$-regular spaces through $\delta\beta$-closures and $\delta\beta$-neighbourhoods, and study their behaviour under irresolute, open, closed and continuous mappings.

(ii) We introduce strongly and strongly * regular spaces, characterize strongly $nCy\mathcal{N}\mathfrak{N}\delta\beta$-regular spaces, and establish the relations among strongly, strongly * and ordinary regular spaces, with examples showing that the converses of some of these relations fail; we also give a non-trivial example of a $\delta\beta$-regular space and show that strong $\delta\beta$-regularity is a very restrictive condition.

(iii) We propose an algorithm, based on the lower and upper approximations and the boundary of an n-Cylindrical neutrosophic evaluation, that identifies the core attributes of nanocarrier formulations and ranks the formulations, we give a condition under which these sets form an n-Cylindrical neutrosophic nano topology, and we show that the resulting space is $\delta\beta$-regular but not regular.

The paper is organized as follows. Section 2 recalls the basic definitions and results. In Section 3, we define n-Cylindrical neutrosophic points, separated sets and the regular spaces, and study their properties. In Section 4, we introduce and study strongly and strongly * regular spaces. In Section 5, we give the application to the selection of nanocarriers in drug delivery technology, with a numerical example. The paper ends with a conclusion.

2 Preliminaries

This section recalls the basic definitions and results that will be used in the subsequent sections.

Definition 2.1 [25] A fuzzy set (briefly, fs) A in U is defined by a membership function $\alpha_A: U \to [0,1]$, where the membership value $\alpha_A(x)$ indicates the degree to which $x \in U$ belongs to the fuzzy set A, for all $x \in U$.

Definition 2.2 [4] A fuzzy topological space (briefly, fts) is a pair (U, τ), where U is a non-empty set and τ is a family of fuzzy sets in U satisfying the following axioms:

(i) $0_U, 1_U \in \tau$, where 0_U and 1_U denote the constant fuzzy sets with membership values 0 and 1, respectively,

(ii) if $A, B \in \tau$, then $A \cap B \in \tau$,

(iii) if $A_i \in \tau$ for each $i \in I$, then $\bigcup_{i\in I} A_i \in \tau$.

Definition 2.3 [2] An intuitionistic fuzzy set (briefly, ifs) A on U is an object of the form

$$A = \{\langle x, \alpha_A(x), \gamma_A(x)\rangle : x \in U\},$$

where $\alpha_A(x) \in [0,1]$ is the degree of positive membership and $\gamma_A(x) \in [0,1]$ is the degree of negative membership of x in A, satisfying $\alpha_A(x) + \gamma_A(x) \leq 1$ for all $x \in U$.

Definition 2.4 [18] A neutrosophic set (briefly, $\mathcal{N}$s) A on U is an object of the form

$$A = \{\langle x, \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle : x \in U\},$$

where $\alpha_A(x)$, $\beta_A(x)$ and $\gamma_A(x) \in [0,1]$ are the degrees of positive, neutral and negative membership of x in A, respectively. The three components are independent of each other, so that $0 \leq \alpha_A(x) + \beta_A(x) + \gamma_A(x) \leq 3$ for all $x \in U$.

Definition 2.5 [14] A neutrosophic topology (briefly, $\mathcal{N}$t) on a non-empty set U is a family τ_N of neutrosophic subsets of U satisfying:

(i) $0_N, 1_N \in \tau_N$,

(ii) $G_1 \cap G_2 \in \tau_N$ for any $G_1, G_2 \in \tau_N$,

(iii) $\bigcup_{i\in J} G_i \in \tau_N$ for any sub-collection $\{G_i : i \in J\} \subseteq \tau_N$.

The pair (U, τ_N) is called a neutrosophic

topological space (briefly, $\mathcal{N}$ts). Any neutrosophic set in τ_N is called a neutrosophic open set (briefly, $\mathcal{N}$os). A neutrosophic set F is neutrosophic closed (briefly, $\mathcal{N}$c) if and only if its complement C(F) is neutrosophic open.

Definition 2.6 [15] An n-Cylindrical neutrosophic set (briefly, nCy$\mathcal{N}$s) A on U is an object of the form

$$A = \{\langle x, \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle : x \in U\},$$

where $\alpha_A(x)$, $\beta_A(x)$ and $\gamma_A(x) \in [0,1]$ are the degrees of positive, neutral and negative membership of x in A, respectively, satisfying

$$0 \le \alpha_A^n(x) + \gamma_A^n(x) \le 1 \quad \text{forall} x \in U,$$

where $n > 1$ is a fixed integer. Here α_A and γ_A are called dependent components, because they are jointly restricted by the condition $\alpha_A^n(x) + \gamma_A^n(x) \le 1$ (a larger positive membership forces a smaller negative membership), whereas β_A is called an independent component, because it may take any value in $[0,1]$ irrespective of α_A and γ_A.

For convenience, the triple $\langle \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle$ is called an n-Cylindrical neutrosophic number (briefly, nCy$\mathcal{N}$N), and the set is written as $A = \langle \alpha_A, \beta_A, \gamma_A\rangle$.

Definition 2.7 [15] Let $\{A_i : i \in I\}$ be an arbitrary family of nCy$\mathcal{N}$s's in U. Then

$$\cap_{i\in I} A_i = \left\{\left\langle x,\ \inf_i(\alpha_{A_i}(x)),\ \inf_i(\beta_{A_i}(x)),\ \sup_i(\gamma_{A_i}(x))\right\rangle : x \in U\right\},$$

$$\cup_{i\in I} A_i = \left\{\left\langle x,\ \sup_i(\alpha_{A_i}(x)),\ \sup_i(\beta_{A_i}(x)),\ \inf_i(\gamma_{A_i}(x))\right\rangle : x \in U\right\}.$$

Definition 2.8 [15] The n-Cylindrical neutrosophic empty and whole sets are defined by

$$0_{nCy\mathcal{N}} = \{\langle x, 0,0,1\rangle : x \in U\}, \quad 1_{nCy\mathcal{N}} = \{\langle x, 1,1,0\rangle : x \in U\}.$$

Definition 2.9 [15] Let CyN(U) denote the family of all nCy$\mathcal{N}$s's on U. The following basic connectives are defined for any $A, B \in CyN(U)$.

(i) **Inclusion.** $A \subseteq B$ if and only if $\alpha_A(x) \le \alpha_B(x)$, $\beta_A(x) \le \beta_B(x)$ and $\gamma_A(x) \ge \gamma_B(x)$ for all $x \in U$. Equality $A = B$ holds when both $A \subseteq B$ and $B \subseteq A$,

(ii) **Union.**

$$A \cup B = \{\langle x,\ \max(\alpha_A(x), \alpha_B(x)),\ \max(\beta_A(x), \beta_B(x)),\ \min(\gamma_A(x), \gamma_B(x))\rangle : x \in U\},$$

(iii) **Intersection.**

$$A \cap B = \{\langle x,\ \min(\alpha_A(x), \alpha_B(x)),\ \min(\beta_A(x), \beta_B(x)),\ \max(\gamma_A(x), \gamma_B(x))\rangle : x \in U\},$$

(iv) **Complement.**

$$A^c = \{\langle x,\ \gamma_A(x),\ 1 - \beta_A(x),\ \alpha_A(x)\rangle : x \in U\}.$$

Remark 2.1 [15] For any nCy$\mathcal{N}$s's A, B, $C \in CyN(U)$ and any family $\{A_i : i \in I\} \subseteq CyN(U)$:

(i) if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$,

(ii) $A \cup B = B \cup A$ and $A \cap B = B \cap A$ (commutativity),

(iii) $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$ (associativity),

(iv) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ and $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ (distributivity),

(v) $A \cap A = A$ and $A \cup A = A$ (idempotency),

(vi) De Morgan's laws: $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$; more generally, $(\cup_{i\in I} A_i)^c = \cap_{i\in I} A_i^c$ and $(\cap_{i\in I} A_i)^c = \cup_{i\in I} A_i^c$,

(vii) $(A^c)^c = A$ (involution),

(viii) if $A \subseteq B$, then $B^c \subseteq A^c$.

Items (vi)–(viii) follow directly from Definitions (ii)7 and (ii)9, since γ and α are interchanged and β is replaced by $1 - \beta$ under complementation.

Remark 2.2 For nCy$\mathcal{N}$s's A and B in U,

(i) $A \cap A^c$ need not be $0_{nCy\mathcal{N}}$,

(ii) $A \cup A^c$ need not be $1_{nCy\mathcal{N}}$,

(iii) the difference of A and B is defined by $A - B = A \cap B^c$, that is,

$$A - B = \{\langle x,\ \min(\alpha_A(x), \gamma_B(x)),\ \min(\beta_A(x), 1 - \beta_B(x)),\ \max(\gamma_A(x), \alpha_B(x))\rangle : x \in U\}.$$

Example 2.1 Let $U = \{a, b\}$ and $n = 3$ Define the nCy$\mathcal{N}$s's

$$A = \{\langle a, 0.9,0.4,0.6\rangle,\ \langle b, 0.5,0.7,0.8\rangle\}, \quad B = \{\langle a, 0.7,0.2,0.8\rangle,\ \langle b, 0.4,0.5,0.9\rangle\}.$$

These are 3-cylindrical neutrosophic sets, since all components lie in [0,1] and

$$(0.9)^3 + (0.6)^3 = 0.945 \le 1, (0.5)^3 + (0.8)^3 = 0.637 \le 1,$$

$$(0.7)^3 + (0.8)^3 = 0.855 \le 1, (0.4)^3 + (0.9)^3 = 0.793 \le 1.$$

Note that $(0.9)^2 + (0.6)^2 = (i)17 > 1$, so A is not a 2-cylindrical neutrosophic set.

By Definition 2.9 (iv),

$$A^c = \{\langle a, 0.6,0.6,0.9\rangle,\ \langle b, 0.8,0.3,0.5\rangle\}, \quad B^c = \{\langle a, 0.8,0.8,0.7\rangle,\ \langle b, 0.9,0.5,0.4\rangle\}.$$

(i) By Definition (ii)9 (iii),

$$A \cap A^c = \{\langle a, \min\{0.9,0.6\}, \min\{0.4,0.6\}, \max\{0.6,0.9\}\rangle,$$

$$\langle b, \min\{0.5,0.8\}, \min\{0.7,0.3\}, \max\{0.8,0.5\}\rangle\} = \{\langle a, 0.6,0.4,0.9\rangle,\ \langle b, 0.5,0.3,0.8\rangle\}.$$

Since, for instance, $\alpha_{A\cap A^c}(a) = 0.6 \ne 0$, we get $A \cap A^c \ne 0_{nCy\mathcal{N}}$.

(ii) By Definition 2.9 (ii),

$$A \cup A^c =$$
$$\{\langle a, \max\{0.9,0.6\}, \max\{0.4,0.6\}, \min\{0.6,0.9\}\rangle,$$

$$\langle b, \max\{0.5,0.8\}, \max\{0.7,0.3\}, \min\{0.8,0.5\}\rangle\}$$
$$= \{\langle a, 0.9,0.6,0.6\rangle,\ \langle b, 0.8,0.7,0.5\rangle\}.$$

Since, for instance, $\alpha_{A\cup A^c}(a) = 0.9 \neq 1$, we get $A \cup A^c \neq 1_{nCy\mathcal{N}}$.

(iii) By Remark 2.2 (iii),

$$A - B = A \cap B^c =$$
$$\{\langle a, \min\{0.9,0.8\}, \min\{0.4,0.8\}, \max\{0.6,0.7\}\rangle,$$

$$\langle b, \min\{0.5,0.9\}, \min\{0.7,0.5\}, \max\{0.8,0.4\}\rangle\}$$
$$= \{\langle a, 0.8,0.4,0.7\rangle,\ \langle b, 0.5,0.5,0.8\rangle\}.$$

This is again a 3-cylindrical neutrosophic set, since $(0.8)^3 + (0.7)^3 = 0.855 \leq 1$ and $(0.5)^3 + (0.8)^3 = 0.637 \leq 1$, and $A - B \subseteq A$.

To illustrate the role of the parameter n, suppose a decision-maker states that an alternative s_i satisfies a criterion C_j to degree 0.85, is undecided about it to degree 0.5, and dissatisfies it to degree 0.6(v) Since $0.85 + 0.65 = (i)5 > 1$ and $0.85^2 + 0.65^2 = (i)145 > 1$, the pair (0.85,0.65) is neither an intuitionistic nor a Pythagorean ($n = 2$) evaluation. However, $0.85^3 + 0.65^3 = 0.88875 \leq 1$, so $\langle 0.85,0.5,0.65\rangle$ is a 3-cylindrical neutrosophic number. Thus larger values of n allow more general evaluations to be modelled.

Definition 2.10 [16] An n-Cylindrical neutrosophic topology (briefly, nCy$\mathcal{N}$t) on a non-empty set U is a family τ_{CyN} of nCy$\mathcal{N}$s's in U satisfying:

(i) $0_{nCy\mathcal{N}}, 1_{nCy\mathcal{N}} \in \tau_{CyN}$,

(ii) $A_1 \cap A_2 \in \tau_{CyN}$ for any $A_1, A_2 \in \tau_{CyN}$,

(iii) $\bigcup_{i\in I} A_i \in \tau_{CyN}$ for any arbitrary sub-collection $\{A_i : i \in I\} \subseteq \tau_{CyN}$.

The pair (U, τ_{CyN}) is called an n-Cylindrical neutrosophic topological space (briefly, nCy$\mathcal{N}$ts). Any nCy$\mathcal{N}$s belonging to τ_{CyN} is called an n-Cylindrical neutrosophic open set (briefly, nCy$\mathcal{N}$os), and its complement is called an n-Cylindrical neutrosophic closed set (briefly, nCy$\mathcal{N}$cs). The family $\{0_{nCy\mathcal{N}}, 1_{nCy\mathcal{N}}\}$ is the indiscrete nCy$\mathcal{N}$t, and the family of all nCy$\mathcal{N}$s's on U is the discrete nCy$\mathcal{N}$t.

Remark 2.3 [16] Every fuzzy topological space, intuitionistic fuzzy topological space and Pythagorean fuzzy topological space can be regarded as an nCy$\mathcal{N}$ts, since the fuzzy, intuitionistic fuzzy and Pythagorean fuzzy sets of those spaces can be viewed as nCy$\mathcal{N}$s's.

Definition 2.11 [16] Let A and B be two nCy$\mathcal{N}$s's in an nCy$\mathcal{N}$ts (U, τ_{CyN}). Then B is called a neighbourhood of A if there exists an nCy$\mathcal{N}$os O such that $A \subseteq O \subseteq B$.

Proposition 2.1 [16] An nCy$\mathcal{N}$s A is an nCy$\mathcal{N}$os in (U, τ_{CyN}) if and only if A is a neighbourhood of every nCy$\mathcal{N}$s contained in it.

Definition 2.12 [16] Let (U, τ_{CyN}) be an nCy$\mathcal{N}$ts and let $A = \{\langle x, \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle : x \in U\}$ be an nCy$\mathcal{N}$s in U. The n-Cylindrical neutrosophic interior (briefly, nCy$\mathcal{N}$int) of A is the union of all nCy$\mathcal{N}$os's contained in A:

$$nCy\mathcal{N}int(A) = \bigcup \{G : G \in \tau_{CyN} \text{and} G \subseteq A\}.$$

Clearly, nCy$\mathcal{N}$int(A) is the largest nCy$\mathcal{N}$os contained in A.

Definition 2.13 [16] Let (U, τ_{CyN}) be an nCy$\mathcal{N}$ts and let $A = \{\langle x, \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle : x \in U\}$ be an nCy$\mathcal{N}$s in U. The n-Cylindrical neutrosophic closure (briefly, nCy$\mathcal{N}$cl) of A is the intersection of all nCy$\mathcal{N}$cs's containing A:

$$nCy\mathcal{N}cl(A) = \bigcap \{K : K^c \in \tau_{CyN} \text{and} A \subseteq K\}.$$

Clearly, nCy$\mathcal{N}$cl(A) is the smallest nCy$\mathcal{N}$cs that contains A.

Definition 2.14 [9] Let U be a non-empty set, R an equivalence relation on U, and A an nCy$\mathcal{N}$s in U with positive membership α_A, neutral membership β_A and negative membership γ_A. The n-Cylindrical neutrosophic lower approximation, n-Cylindrical neutrosophic upper approximation and n-Cylindrical neutrosophic boundary of A in (U, R), denoted by $\underline{nCy\mathcal{N}}(A)$, $\overline{nCy\mathcal{N}}(A)$ and $B_{nCy\mathcal{N}}(A)$, respectively, are defined as follows:

(i) $\underline{nCy\mathcal{N}}(A) = \{\langle x,\ \alpha_{\underline{R}(A)}(x),\ \beta_{\underline{R}(A)}(x),\ \gamma_{\underline{R}(A)}(x)\rangle : x \in U\}$,

(ii) $\overline{nCy\mathcal{N}}(A) = \left\{\left\langle x,\ \alpha_{\overline{R}(A)}(x),\ \beta_{\overline{R}(A)}(x),\ \gamma_{\overline{R}(A)}(x)\right\rangle : x \in U\right\}$,

(iii) $B_{nCy\mathcal{N}}(A) = \overline{nCy\mathcal{N}}(A) - \underline{nCy\mathcal{N}}(A) = \overline{nCy\mathcal{N}}(A) \cap (\underline{nCy\mathcal{N}}(A))^c$,

where the component functions are given by

$$\alpha_{\underline{R}(A)}(x) = \bigwedge_{y\in[x]_R} \alpha_A(y), \beta_{\underline{R}(A)}(x) = \bigwedge_{y\in[x]_R} \beta_A(y), \gamma_{\underline{R}(A)}(x) = \bigvee_{y\in[x]_R} \gamma_A(y),$$

$$\alpha_{\overline{R}(A)}(x) = \bigvee_{y\in[x]_R} \alpha_A(y), \beta_{\overline{R}(A)}(x) = \bigvee_{y\in[x]_R} \beta_A(y), \gamma_{\overline{R}(A)}(x) = \bigwedge_{y\in[x]_R} \gamma_A(y).$$

Here $[x]_R$ denotes the equivalence class of x under R.

Definition 2.15 [9] Let U be a universe, R an equivalence relation on U, and F an nCy$\mathcal{N}$s in U. If the collection

$$\tau_R^{nCy\mathcal{N}}(F) = \{\ 0_{nCy\mathcal{N}},\ 1_{nCy\mathcal{N}},\ \underline{nCy\mathcal{N}}(F),\ \overline{nCy\mathcal{N}}(F),\ B_{nCy\mathcal{N}}(F)\ \}$$

forms an nCy$\mathcal{N}$t on U, then it is called the n-Cylindrical

neutrosophic nano topology on U with respect to F. The pair $(U, \tau_R^{nCy\mathcal{N}}(F))$ is called an n-Cylindrical neutrosophic nano topological space (briefly, nCy$\mathcal{N}\mathfrak{N}$ts). The elements of $\tau_R^{nCy\mathcal{N}}(F)$ are called n-Cylindrical neutrosophic nano open sets (briefly, nCy$\mathcal{N}\mathfrak{N}$os), and their complements are called n-Cylindrical neutrosophic nano closed sets (briefly, nCy$\mathcal{N}\mathfrak{N}$cs).

Definition 2.16 [9] Let $(U, \tau_R^{nCy\mathcal{N}}(F))$ be an nCy$\mathcal{N}\mathfrak{N}$ts with respect to F, where F is an nCy$\mathcal{N}$s in U, and let S be an nCy$\mathcal{N}$s in U. Then the n-Cylindrical neutrosophic nano

(i) interior of S (briefly, nCy$\mathcal{N}\mathfrak{N}$int(S)) is defined by

$$nCy\mathcal{N}\mathfrak{N}int(S) = \cup \{I: I \subseteq S \ \& \ IisannCy\mathcal{N}\mathfrak{N}osetinU\},$$

(ii) closure of S (briefly, nCy$\mathcal{N}\mathfrak{N}$cl(S)) is defined by

$$nCy\mathcal{N}\mathfrak{N}cl(S) = \cap \{K: S \subseteq K \ \& \ KisannCy\mathcal{N}\mathfrak{N}csetinU\}.$$

3 n-Cylindrical neutrosophic nano (resp. δ, δ pre, δ semi, δα and δβ) regular spaces

In this section, we extend the study of n-Cylindrical neutrosophic nano topological spaces by introducing n-Cylindrical neutrosophic nano regular spaces, along with their variantsâ€”namely δ, δ pre, δ semi, δα and δβ-regular spaces. These spaces are defined using the corresponding types of n-Cylindrical neutrosophic nano open and closed sets.

Definition 3.1 Let $(U, \tau_R^{nCy\mathcal{N}}(F))$ be an nCy$\mathcal{N}\mathfrak{N}$ts and let S be an nCy$\mathcal{N}$s in U. Then S is called an n-Cylindrical neutrosophic nano

(i) regular open set (briefly, nCy$\mathcal{N}\mathfrak{N}$ro set) if $S = nCy\mathcal{N}\mathfrak{N}int(nCy\mathcal{N}\mathfrak{N}cl(S))$,

(ii) regular closed set (briefly, nCy$\mathcal{N}\mathfrak{N}$rc set) if $S = nCy\mathcal{N}\mathfrak{N}cl(nCy\mathcal{N}\mathfrak{N}int(S))$.

Since $(nCy\mathcal{N}\mathfrak{N}int(S))^c = nCy\mathcal{N}\mathfrak{N}cl(S^c)$ and $(nCy\mathcal{N}\mathfrak{N}cl(S))^c = nCy\mathcal{N}\mathfrak{N}int(S^c)$ (by Remark 2.1 (vi)–(viii)), S is an nCy$\mathcal{N}\mathfrak{N}$ro set if and only if S^c is an nCy$\mathcal{N}\mathfrak{N}$rc set.

Definition 3.2 Let $(U, \tau_R^{nCy\mathcal{N}}(F))$ be an nCy$\mathcal{N}\mathfrak{N}$ts and let S be an nCy$\mathcal{N}$s in U. Then the n-Cylindrical neutrosophic nano

(i) δ-interior of S (briefly, nCy$\mathcal{N}\mathfrak{N}\delta$int(S)) is defined by

$$nCy\mathcal{N}\mathfrak{N}\delta int(S) = \cup \{I: I \subseteq S \ \& \ IisannCy\mathcal{N}\mathfrak{N}rosetinU\},$$

(ii) δ-closure of S (briefly, nCy$\mathcal{N}\mathfrak{N}\delta$cl(S)) is defined by

$$nCy\mathcal{N}\mathfrak{N}\delta cl(S) = \cap \{K: S \subseteq K \ \& \ KisannCy\mathcal{N}\mathfrak{N}rcsetinU\}.$$

Definition 3.3 Let $(U, \tau_R^{nCy\mathcal{N}}(F))$ be an nCy$\mathcal{N}\mathfrak{N}$ts. Then an nCy$\mathcal{N}$s S in U is said to be an n-Cylindrical neutrosophic nano

(i) open set (briefly, nCy$\mathcal{N}\mathfrak{N}$os) if $S = nCy\mathcal{N}\mathfrak{N}int(S)$, i.e., $S \in \tau_R^{nCy\mathcal{N}}(F)$,

(ii) δ-open set (briefly, nCy$\mathcal{N}\mathfrak{N}\delta$os) if $S = nCy\mathcal{N}\mathfrak{N}\delta int(S)$,

(iii) δ-pre open set (briefly, nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$os) if $S \subseteq nCy\mathcal{N}\mathfrak{N}int(nCy\mathcal{N}\mathfrak{N}\delta cl(S))$,

(iv) δ-semi open set (briefly, nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{S}$os) if $S \subseteq nCy\mathcal{N}\mathfrak{N}cl(nCy\mathcal{N}\mathfrak{N}\delta int(S))$,

(v) δα-open set (briefly, nCy$\mathcal{N}\mathfrak{N}\delta\alpha$os) if $S \subseteq nCy\mathcal{N}\mathfrak{N}int(nCy\mathcal{N}\mathfrak{N}cl(nCy\mathcal{N}\mathfrak{N}\delta int(S)))$,

(vi) δβ-open set (briefly, nCy$\mathcal{N}\mathfrak{N}\delta\beta$os) if $S \subseteq nCy\mathcal{N}\mathfrak{N}cl(nCy\mathcal{N}\mathfrak{N}int(nCy\mathcal{N}\mathfrak{N}\delta cl(S)))$.

The complement of an nCy$\mathcal{N}\mathfrak{N}$os (resp. nCy$\mathcal{N}\mathfrak{N}\delta$os, nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}os, nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}os, nCy\mathcal{N}\mathfrak{N}\delta\alpha$os and nCy$\mathcal{N}\mathfrak{N}\delta\beta$os) is called an n-Cylindrical neutrosophic nano (resp. δ-, δ-pre, δ-semi, δα- and δβ-) closed set (briefly, nCy$\mathcal{N}\mathfrak{N}$cs (resp. nCy$\mathcal{N}\mathfrak{N}\delta$cs, nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}cs, nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}cs, nCy\mathcal{N}\mathfrak{N}\delta\alpha$cs and nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs)) in U. For convenience, we also say that S is an nCy$\mathcal{N}\mathfrak{N}\delta$o (resp. nCy$\mathcal{N}\mathfrak{N}\delta$c) set if it is an nCy$\mathcal{N}\mathfrak{N}\delta$os (resp. nCy$\mathcal{N}\mathfrak{N}\delta$cs), and similarly for $\delta\mathcal{P}$, $\delta\mathcal{S}$, δα and δβ.

Definition 3.4 Let $(U, \tau_R^{nCy\mathcal{N}}(F))$ be an nCy$\mathcal{N}\mathfrak{N}$ts and let S be an nCy$\mathcal{N}$s in U. Then the n-Cylindrical neutrosophic nano

(i) δ-pre interior of S (briefly, nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$int(S)) is defined by

$$nCy\mathcal{N}\mathfrak{N}\delta\mathcal{P}int(S) = \cup \{I: I \subseteq S \ \& \ IisannCy\mathcal{N}\mathfrak{N}\delta\mathcal{P}osetinU\},$$

(ii) δ-semi interior of S (briefly, nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{S}$int(S)) is defined by

$$nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}int(S) = \cup \{I: I \subseteq S \ \& \ IisannCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}osetinU\},$$

(iii) δα-interior of S (briefly, nCy$\mathcal{N}\mathfrak{N}\delta\alpha$int(S)) is defined by

$$nCy\mathcal{N}\mathfrak{N}\delta\alpha int(S) = \cup \{I: I \subseteq S \ \& \ IisannCy\mathcal{N}\mathfrak{N}\delta\alpha osetinU\},$$

(iv) δβ-interior of S (briefly, nCy$\mathcal{N}\mathfrak{N}\delta\beta$int(S)) is defined by

$$nCy\mathcal{N}\mathfrak{N}\delta\beta int(S) = \cup \{I: I \subseteq S \ \& \ IisannCy\mathcal{N}\mathfrak{N}\delta\beta osetinU\},$$

(v) δ-pre closure of S (briefly, nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$cl(S)) is defined by

$$nCy\mathcal{N}\mathfrak{N}\delta\mathcal{P}cl(S) = \cap \{K: S \subseteq K \ \& \ KisannCy\mathcal{N}\mathfrak{N}\delta\mathcal{P}csetinU\},$$

(vi) δ-semi closure of S (briefly, nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{S}$cl(S)) is defined by

$$nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}cl(S) = \cap \{K: S \subseteq K \ \& \ KisannCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}csetinU\},$$

(vii) δα-closure of S (briefly, nCy$\mathcal{N}\mathfrak{N}\delta\alpha$cl(S))

is defined by

$$nCy\mathcal{N}\mathfrak{N}\delta\alpha cl(S) = \bigcap \{K: S \subseteq K \ \& \ K is a nCy\mathcal{N}\mathfrak{N}\delta\alpha cset in U\},$$

(viii) $\delta\beta$-closure of S (briefly, $nCy\mathcal{N}\mathfrak{N}\delta\beta cl(S)$) is defined by

$$nCy\mathcal{N}\mathfrak{N}\delta\beta cl(S) = \bigcap \{K: S \subseteq K \ \& \ K is a nCy\mathcal{N}\mathfrak{N}\delta\beta cset in U\}.$$

Definition 3.5 Let $h_{\mathcal{C}}: U_1 \to U_2$ be a function.

(i) For an $nCy\mathcal{N}s$ $T = \{\langle y, \alpha_T(y), \beta_T(y), \gamma_T(y)\rangle : y \in U_2\}$ in U_2, the inverse image of T is the $nCy\mathcal{N}s$

$$h_{\mathcal{C}}^{-1}(T) = \{\langle x,\ \alpha_T(h_{\mathcal{C}}(x)),\ \beta_T(h_{\mathcal{C}}(x)),\ \gamma_T(h_{\mathcal{C}}(x))\rangle : x \in U_1\}.$$

(ii) For an $nCy\mathcal{N}s$ $S = \{\langle x, \alpha_S(x), \beta_S(x), \gamma_S(x)\rangle : x \in U_1\}$ in U_1, the image of S is the $nCy\mathcal{N}s$ $h_{\mathcal{C}}(S) = \{\langle y, \alpha_{h_{\mathcal{C}}(S)}(y), \beta_{h_{\mathcal{C}}(S)}(y), \gamma_{h_{\mathcal{C}}(S)}(y)\rangle : y \in U_2\}$, where

$$(\alpha_{h_{\mathcal{C}}(S)}(y), \beta_{h_{\mathcal{C}}(S)}(y), \gamma_{h_{\mathcal{C}}(S)}(y)) = \begin{cases} (\sup_{x\in X_y} \alpha_S(x),\ \sup_{x\in X_y} \beta_S(x),\ \inf_{x\in X_y} \gamma_S(x)), & \text{if } X_y \neq \emptyset, \\ (0,0,1), & \text{otherwise.} \end{cases}$$

where $X_y = h_{\mathcal{C}}^{-1}(y) = \{x \in U_1 : h_{\mathcal{C}}(x) = y\}$.

Clearly $h_{\mathcal{C}}^{-1}(T)$ is an $nCy\mathcal{N}s$.

$h_{\mathcal{C}}(S)$ is also an $nCy\mathcal{N}s$. Let $y \in U_2$ with $h_{\mathcal{C}}^{-1}(y) \neq \emptyset$, let $\varepsilon > 0$, and choose $x_0 \in h_{\mathcal{C}}^{-1}(y)$ with $\alpha_S(x_0) > \alpha_{h_{\mathcal{C}}(S)}(y) - \varepsilon$. Since $\gamma_{h_{\mathcal{C}}(S)}(y) \leq \gamma_S(x_0)$,

$$(\alpha_{h_{\mathcal{C}}(S)}(y) - \varepsilon)^n + \gamma_{h_{\mathcal{C}}(S)}(y)^n \leq \alpha_S(x_0)^n + \gamma_S(x_0)^n \leq 1.$$

Letting $\varepsilon \to 0$ gives $\alpha_{h_{\mathcal{C}}(S)}(y)^n + \gamma_{h_{\mathcal{C}}(S)}(y)^n \leq 1$.

Definition 3.6 Let $(U_1, \tau_R^{nCy\mathcal{N}}(A_1))$ and $(U_2, \tau_R^{nCy\mathcal{N}}(A_2))$ be two $nCy\mathcal{N}\mathfrak{N}ts$'s. Then a function $h_{\mathcal{C}}: (U_1, \tau_R^{nCy\mathcal{N}}(A_1)) \to (U_2, \tau_R^{nCy\mathcal{N}}(A_2))$ is said to be a n-Cylindrical neutrosophic nano (resp. δ, δ pre, δ semi, $\delta\alpha$ and $\delta\beta$) continuous (briefly, $nCy\mathcal{N}\mathfrak{N}Cts$ (resp. $nCy\mathcal{N}\mathfrak{N}\delta Cts$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{P}Cts$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}Cts$, $nCy\mathcal{N}\mathfrak{N}\delta\alpha Cts$ and $nCy\mathcal{N}\mathfrak{N}\delta\beta Cts$)) function if $h_{\mathcal{C}}^{-1}(G)$ is $nCy\mathcal{N}\mathfrak{N}o$ (resp. $nCy\mathcal{N}\mathfrak{N}\delta o$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{P}o$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}o$, $nCy\mathcal{N}\mathfrak{N}\delta\alpha o$ & $nCy\mathcal{N}\mathfrak{N}\delta\beta o$) set in U_1 for all $nCy\mathcal{N}\mathfrak{N}o$ set G in U_2.

Definition 3.7 Let $(U_1, \tau_R^{nCy\mathcal{N}}(A_1))$ and $(U_2, \tau_R^{nCy\mathcal{N}}(A_2))$ be two $nCy\mathcal{N}\mathfrak{N}ts$'s. A function $h_{\mathcal{C}}: (U_1, \tau_R^{nCy\mathcal{N}}(A_1)) \to (U_2, \tau_R^{nCy\mathcal{N}}(A_2))$ is said to be n-Cylindrical neutrosophic nano (resp. δ, δ pre, δ semi, $\delta\alpha$ and $\delta\beta$) open (briefly, $nCy\mathcal{N}\mathfrak{N}O$ (resp. $nCy\mathcal{N}\mathfrak{N}\delta O$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{P}O$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}O$, $nCy\mathcal{N}\mathfrak{N}\delta\alpha O$ and $nCy\mathcal{N}\mathfrak{N}\delta\beta O$)) if $h_{\mathcal{C}}(G)$ is an $nCy\mathcal{N}\mathfrak{N}os$ (resp. $nCy\mathcal{N}\mathfrak{N}\delta os$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{P}os$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}os$, $nCy\mathcal{N}\mathfrak{N}\delta\alpha os$ and $nCy\mathcal{N}\mathfrak{N}\delta\beta os$) in U_2 for every $nCy\mathcal{N}\mathfrak{N}os$ G in U_1, where $h_{\mathcal{C}}(G)$ is the image of G in the sense of Definition 3.5

Definition 3.8 Let $(U, \tau_R^{nCy\mathcal{N}}(F))$ be an $nCy\mathcal{N}\mathfrak{N}ts$.

(i) For $x \in U$ and $p = \langle a, b, c\rangle$ with $a, b, c \in [0,1]$ and $a^n + c^n \leq 1$, the n-Cylindrical neutrosophic point x_p is the $nCy\mathcal{N}s$ in U whose value at x is $\langle a, b, c\rangle$ and whose value at every other point of U is $\langle 0,0,1\rangle$. For an $nCy\mathcal{N}s$ A in U, we write $x_p \in A$ if $x_p \subseteq A$, that is, $a \leq \alpha_A(x)$, $b \leq \beta_A(x)$ and $c \geq \gamma_A(x)$. For an $nCy\mathcal{N}s$ S in U and $x \in U$, we write $x_{S(x)}$ for the n-Cylindrical neutrosophic point with $S(x) = \langle \alpha_S(x), \beta_S(x), \gamma_S(x)\rangle$. Then $x_{S(x)} \in S$ for every $x \in U$, and $S = \bigcup_{x\in U} x_{S(x)}$.

(ii) Two $nCy\mathcal{N}s$'s L and M in U are said to be separated if $L \subseteq M^c$ (equivalently, $M \subseteq L^c$).

(iii) An $nCy\mathcal{N}s$ N in U is called an n-Cylindrical neutrosophic nano $\delta\beta$-neighbourhood (briefly, $nCy\mathcal{N}\mathfrak{N}\delta\beta nbhd$) of an $nCy\mathcal{N}s$ K if there exists an $nCy\mathcal{N}\mathfrak{N}\delta\beta os$ V such that $K \subseteq V \subseteq N$.

(iv) $(U, \tau_R^{nCy\mathcal{N}}(F))$ is said to be n-Cylindrical neutrosophic nano (resp. δ, δ pre, δ semi, $\delta\alpha$ and $\delta\beta$) regular (briefly, $nCy\mathcal{N}\mathfrak{N}Reg$ (resp. $nCy\mathcal{N}\mathfrak{N}\delta Reg$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{P}Reg$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}Reg$, $nCy\mathcal{N}\mathfrak{N}\delta\alpha Reg$ and $nCy\mathcal{N}\mathfrak{N}\delta\beta Reg$)) if for every $nCy\mathcal{N}\mathfrak{N}cs$ (resp. $nCy\mathcal{N}\mathfrak{N}\delta cs$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{P}cs$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}cs$, $nCy\mathcal{N}\mathfrak{N}\delta\alpha cs$ and $nCy\mathcal{N}\mathfrak{N}\delta\beta cs$) A in U and every n-Cylindrical neutrosophic point $x_p \in A^c$, there exist separated $nCy\mathcal{N}\mathfrak{N}os$'s (resp. $nCy\mathcal{N}\mathfrak{N}\delta os$'s, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{P}os$'s, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}os$'s, $nCy\mathcal{N}\mathfrak{N}\delta\alpha os$'s and $nCy\mathcal{N}\mathfrak{N}\delta\beta os$'s) L and M in U such that $A \subseteq L$ and $x_p \in M$.

If $L \cap M = 0_{nCy\mathcal{N}}$, then L and M are separated (see the proof of Theorem 3.1); the converse need not hold, since $S \cap S^c$ need not be $0_{nCy\mathcal{N}}$.

Theorem 3.1 In an $nCy\mathcal{N}\mathfrak{N}ts$ $(U, \tau_R^{nCy\mathcal{N}}(F))$, the following statements are equivalent:

(i) U is $nCy\mathcal{N}\mathfrak{N}\delta\beta Reg$;

(ii) for every $nCy\mathcal{N}\mathfrak{N}\delta\beta os$ G in U and every n-Cylindrical neutrosophic point $x_p \in G$, there exists an $nCy\mathcal{N}\mathfrak{N}\delta\beta os$ L such that $x_p \in L \subseteq nCy\mathcal{N}\mathfrak{N}\delta\beta cl(L) \subseteq G$;

(iii) for every $nCy\mathcal{N}\mathfrak{N}\delta\beta cs$ K in U and every n-Cylindrical neutrosophic point $x_p \in K^c$, there exists an $nCy\mathcal{N}\mathfrak{N}\delta\beta cs$ N which is an $nCy\mathcal{N}\mathfrak{N}\delta\beta nbhd$ of K such that $x_p \in N^c$.

Proof. We use the following facts. For $nCy\mathcal{N}s$'s S and T, $(S^c)^c = S$, and $S \subseteq T$ implies $T^c \subseteq S^c$. An $nCy\mathcal{N}s$ S is an $nCy\mathcal{N}\mathfrak{N}\delta\beta os$ if and only if S^c is an $nCy\mathcal{N}\mathfrak{N}\delta\beta cs$, and $nCy\mathcal{N}\mathfrak{N}\delta\beta cl(S)$ is the smallest $nCy\mathcal{N}\mathfrak{N}\delta\beta cs$ containing S (any intersection of $nCy\mathcal{N}\mathfrak{N}\delta\beta cs$'s is an $nCy\mathcal{N}\mathfrak{N}\delta\beta cs$, because any union of $nCy\mathcal{N}\mathfrak{N}\delta\beta os$'s is an $nCy\mathcal{N}\mathfrak{N}\delta\beta os$). Moreover, if $L \cap M = 0_{nCy\mathcal{N}}$, then $M \subseteq L^c$: at each $x \in U$, one of $\gamma_L(x)$, $\gamma_M(x)$ equals 1, and $\gamma(x) = 1$ forces $\alpha(x) = 0$ by $\alpha^n +$

$\gamma^n \leq 1$, while one of $\beta_L(x)$, $\beta_M(x)$ equals 0; hence $\alpha_M(x) \leq \gamma_L(x)$, $\beta_M(x) \leq 1 - \beta_L(x)$ and $\gamma_M(x) \geq \alpha_L(x)$.

(i) ⇒ (ii): Suppose that U is nCy$\mathcal{N}\mathfrak{N}$δβReg. Let G be an nCy$\mathcal{N}\mathfrak{N}$δβos in U and let $x_p \in G$. Then $A = G^c$ is an nCy$\mathcal{N}\mathfrak{N}$δβcs and $x_p \in G = A^c$. Since U is nCy$\mathcal{N}\mathfrak{N}$δβReg, there exist separated nCy$\mathcal{N}\mathfrak{N}$δβos's M and L such that $A \subseteq M$ and $x_p \in L$. Since $L \subseteq M^c$ and M^c is an nCy$\mathcal{N}\mathfrak{N}$δβcs, we get nCy$\mathcal{N}\mathfrak{N}$δβcl(L) $\subseteq M^c$. Since $G^c = A \subseteq M$, we have $M^c \subseteq G$. Hence $x_p \in L \subseteq$ nCy$\mathcal{N}\mathfrak{N}$δβcl(L) $\subseteq G$.

(ii) ⇒ (iii): Let K be an nCy$\mathcal{N}\mathfrak{N}$δβcs in U and let $x_p \in K^c$. Since K^c is an nCy$\mathcal{N}\mathfrak{N}$δβos, by (ii) there exists an nCy$\mathcal{N}\mathfrak{N}$δβos L such that $x_p \in L \subseteq$ nCy$\mathcal{N}\mathfrak{N}$δβcl(L) $\subseteq K^c$. Put $N = L^c$ and $V = (\text{nCy}\mathcal{N}\mathfrak{N}\delta\beta\text{cl(L)})^c$. Then N is an nCy$\mathcal{N}\mathfrak{N}$δβcs, V is an nCy$\mathcal{N}\mathfrak{N}$δβos, and taking complements in $L \subseteq$ nCy$\mathcal{N}\mathfrak{N}$δβcl(L) $\subseteq K^c$ gives $K \subseteq V \subseteq N$. Hence N is an nCy$\mathcal{N}\mathfrak{N}$δβnbhd of K, and $x_p \in L = N^c$.

(iii) ⇒ (i): Let A be an nCy$\mathcal{N}\mathfrak{N}$δβcs in U and let $x_p \in A^c$. By (iii), there exist an nCy$\mathcal{N}\mathfrak{N}$δβcs N and an nCy$\mathcal{N}\mathfrak{N}$δβos V such that $A \subseteq V \subseteq N$ and $x_p \in N^c$. Put $L = V$ and $M = N^c$. Then L and M are nCy$\mathcal{N}\mathfrak{N}$δβos's, $A \subseteq L$ and $x_p \in M$. Since $V \subseteq N$, we have $L = V \subseteq N = (N^c)^c = M^c$, so L and M are separated. Hence U is nCy$\mathcal{N}\mathfrak{N}$δβReg.

Corollary 3.1 In an nCy$\mathcal{N}\mathfrak{N}$ts $(U, \tau_R^{\text{nCy}\mathcal{N}}(F))$, the following statements are equivalent:

(i) U is nCy$\mathcal{N}\mathfrak{N}$Reg;

(ii) for every nCy$\mathcal{N}\mathfrak{N}$os G in U and every n-Cylindrical neutrosophic point $x_p \in G$, there exists an nCy$\mathcal{N}\mathfrak{N}$os L such that $x_p \in L \subseteq$ nCy$\mathcal{N}\mathfrak{N}$cl(L) $\subseteq G$;

(iii) for every nCy$\mathcal{N}\mathfrak{N}$cs K in U and every n-Cylindrical neutrosophic point $x_p \in K^c$, there exist an nCy$\mathcal{N}\mathfrak{N}$cs N and an nCy$\mathcal{N}\mathfrak{N}$os V such that $K \subseteq V \subseteq N$ and $x_p \in N^c$.

Moreover, if U is nCy$\mathcal{N}\mathfrak{N}$Reg, then for every nCy$\mathcal{N}\mathfrak{N}$os G in U and every n-Cylindrical neutrosophic point $x_p \in G$, there exists an nCy$\mathcal{N}\mathfrak{N}$δβos L such that $x_p \in L \subseteq$ nCy$\mathcal{N}\mathfrak{N}$δβcl(L) $\subseteq G$.

Proof. The equivalence of (i), (ii) and (iii) is proved exactly as in Theorem 3.1, with nCy$\mathcal{N}\mathfrak{N}$δβos's, nCy$\mathcal{N}\mathfrak{N}$δβcs's and nCy$\mathcal{N}\mathfrak{N}$δβcl replaced by nCy$\mathcal{N}\mathfrak{N}$os's, nCy$\mathcal{N}\mathfrak{N}$cs's and nCy$\mathcal{N}\mathfrak{N}$cl, respectively; note that nCy$\mathcal{N}\mathfrak{N}$cl(S) is the smallest nCy$\mathcal{N}\mathfrak{N}$cs containing S.

For the last statement, let U be nCy$\mathcal{N}\mathfrak{N}$Reg, let G be an nCy$\mathcal{N}\mathfrak{N}$os and let $x_p \in G$. By (ii), there exists an nCy$\mathcal{N}\mathfrak{N}$os L such that $x_p \in L \subseteq$ nCy$\mathcal{N}\mathfrak{N}$cl(L) $\subseteq G$. Since $L = \text{nCy}\mathcal{N}\mathfrak{N}\text{int(L)} \subseteq \text{nCy}\mathcal{N}\mathfrak{N}\text{int}(\text{nCy}\mathcal{N}\mathfrak{N}\delta\text{cl(L)}) \subseteq \text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(\text{nCy}\mathcal{N}\mathfrak{N}\text{int}(\text{nCy}\mathcal{N}\mathfrak{N}\delta\text{cl(L)}))$, L is an nCy$\mathcal{N}\mathfrak{N}$δβos. Taking complements in the same argument, every nCy$\mathcal{N}\mathfrak{N}$cs is an nCy$\mathcal{N}\mathfrak{N}$δβcs; in particular, nCy$\mathcal{N}\mathfrak{N}$cl(L) is an nCy$\mathcal{N}\mathfrak{N}$δβcs containing L, and so nCy$\mathcal{N}\mathfrak{N}$δβcl(L) $\subseteq$ nCy$\mathcal{N}\mathfrak{N}$cl(L). Hence $x_p \in L \subseteq$ nCy$\mathcal{N}\mathfrak{N}$δβcl(L) $\subseteq G$.

Theorem 3.2 An nCy$\mathcal{N}\mathfrak{N}$ts $(U, \tau_R^{\text{nCy}\mathcal{N}}(F))$ is nCy$\mathcal{N}\mathfrak{N}$δβReg if and only if for every n-Cylindrical neutrosophic point x_p in U and every nCy$\mathcal{N}\mathfrak{N}$δβnbhd N of x_p, there exists an nCy$\mathcal{N}\mathfrak{N}$δβos M such that $x_p \in M \subseteq$ nCy$\mathcal{N}\mathfrak{N}$δβcl(M) $\subseteq$ N.

Proof. Necessity. Let U be nCy$\mathcal{N}\mathfrak{N}$δβReg, let x_p be an n-Cylindrical neutrosophic point in U and let N be an nCy$\mathcal{N}\mathfrak{N}$δβnbhd of x_p. Then there exists an nCy$\mathcal{N}\mathfrak{N}$δβos G such that $x_p \subseteq G \subseteq N$, that is, $x_p \in G \subseteq N$. Now G^c is an nCy$\mathcal{N}\mathfrak{N}$δβcs and $x_p \in G = (G^c)^c$. Since U is nCy$\mathcal{N}\mathfrak{N}$δβReg, there exist separated nCy$\mathcal{N}\mathfrak{N}$δβos's L and M such that $G^c \subseteq L$ and $x_p \in M$. Since $M \subseteq L^c$ and L^c is an nCy$\mathcal{N}\mathfrak{N}$δβcs, we get nCy$\mathcal{N}\mathfrak{N}$δβcl(M) $\subseteq L^c$. Also, $G^c \subseteq L$ implies $L^c \subseteq G \subseteq N$. Hence $x_p \in M \subseteq$ nCy$\mathcal{N}\mathfrak{N}$δβcl(M) $\subseteq$ N.

Sufficiency. Suppose that the condition holds. Let A be an nCy$\mathcal{N}\mathfrak{N}$δβcs in U and let $x_p \in A^c$. Since A^c is an nCy$\mathcal{N}\mathfrak{N}$δβos and $x_p \subseteq A^c \subseteq A^c$, A^c is an nCy$\mathcal{N}\mathfrak{N}$δβnbhd of x_p. By hypothesis, there exists an nCy$\mathcal{N}\mathfrak{N}$δβos M such that $x_p \in M \subseteq$ nCy$\mathcal{N}\mathfrak{N}$δβcl(M) $\subseteq A^c$. Put $L = (\text{nCy}\mathcal{N}\mathfrak{N}\delta\beta\text{cl(M)})^c$. Since nCy$\mathcal{N}\mathfrak{N}$δβcl(M) is an nCy$\mathcal{N}\mathfrak{N}$δβcs, L is an nCy$\mathcal{N}\mathfrak{N}$δβos, and taking complements in nCy$\mathcal{N}\mathfrak{N}$δβcl(M) $\subseteq A^c$ gives $A \subseteq L$. Moreover, $M \subseteq \text{nCy}\mathcal{N}\mathfrak{N}\delta\beta\text{cl(M)} = L^c$, so L and M are separated. Thus L and M are separated nCy$\mathcal{N}\mathfrak{N}$δβos's with $A \subseteq L$ and $x_p \in M$. Hence U is nCy$\mathcal{N}\mathfrak{N}$δβReg.

The proofs for nCy$\mathcal{N}\mathfrak{N}$Reg, nCy$\mathcal{N}\mathfrak{N}$δReg, nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}Reg, nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}$Reg and nCy$\mathcal{N}\mathfrak{N}$δαReg spaces are similar.

Theorem 3.3 An nCy$\mathcal{N}\mathfrak{N}$ts $(U, \tau_R^{\text{nCy}\mathcal{N}}(F))$ is nCy$\mathcal{N}\mathfrak{N}$δβReg if and only if for every nCy$\mathcal{N}\mathfrak{N}$δβcs A in U and every n-Cylindrical neutrosophic point $x_p \in A^c$, there exist nCy$\mathcal{N}\mathfrak{N}$δβos's L and M in U such that $x_p \in L$, $A \subseteq M$ and nCy$\mathcal{N}\mathfrak{N}$δβcl(L) and nCy$\mathcal{N}\mathfrak{N}$δβcl(M) are separated, that is, nCy$\mathcal{N}\mathfrak{N}$δβcl(L) $\subseteq (\text{nCy}\mathcal{N}\mathfrak{N}\delta\beta\text{cl(M)})^c$.

Proof. Necessity. Let U be nCy$\mathcal{N}\mathfrak{N}$δβReg, let A be an nCy$\mathcal{N}\mathfrak{N}$δβcs in U and let $x_p \in A^c$. Then there exist separated nCy$\mathcal{N}\mathfrak{N}$δβos's M and L_1 such that $A \subseteq M$ and $x_p \in L_1$. Since $M \subseteq L_1^c$ and L_1^c is an nCy$\mathcal{N}\mathfrak{N}$δβcs, we get nCy$\mathcal{N}\mathfrak{N}$δβcl(M) $\subseteq L_1^c$, and so $x_p \in L_1 \subseteq (\text{nCy}\mathcal{N}\mathfrak{N}\delta\beta\text{cl(M)})^c$. Now nCy$\mathcal{N}\mathfrak{N}$δβcl(M) is an nCy$\mathcal{N}\mathfrak{N}$δβcs and $x_p \in (\text{nCy}\mathcal{N}\mathfrak{N}\delta\beta\text{cl(M)})^c$. Since U is nCy$\mathcal{N}\mathfrak{N}$δβReg, there exist separated nCy$\mathcal{N}\mathfrak{N}$δβos's H and G such that nCy$\mathcal{N}\mathfrak{N}$δβcl(M) $\subseteq$ H and $x_p \in G$. Since $G \subseteq H^c$ and H^c is an nCy$\mathcal{N}\mathfrak{N}$δβcs, we get nCy$\mathcal{N}\mathfrak{N}$δβcl(G) $\subseteq H^c$. Also, nCy$\mathcal{N}\mathfrak{N}$δβcl(M) $\subseteq$ H implies $H^c \subseteq (\text{nCy}\mathcal{N}\mathfrak{N}\delta\beta\text{cl(M)})^c$. Take $L = G$. Then L and M are nCy$\mathcal{N}\mathfrak{N}$δβos's such that $x_p \in L$, $A \subseteq M$ and

$nCy\mathcal{N}\mathfrak{N}\delta\beta cl(L) \subseteq H^c \subseteq (nCy\mathcal{N}\mathfrak{N}\delta\beta cl(M))^c$, that is, $nCy\mathcal{N}\mathfrak{N}\delta\beta cl(L)$ and $nCy\mathcal{N}\mathfrak{N}\delta\beta cl(M)$ are separated.

Sufficiency. Suppose that the condition holds. Let A be an $nCy\mathcal{N}\mathfrak{N}\delta\beta cs$ in U and let $x_p \in A^c$. Then there exist $nCy\mathcal{N}\mathfrak{N}\delta\beta os$'s L and M such that $x_p \in L$, $A \subseteq M$ and $nCy\mathcal{N}\mathfrak{N}\delta\beta cl(L) \subseteq (nCy\mathcal{N}\mathfrak{N}\delta\beta cl(M))^c$. Since $M \subseteq nCy\mathcal{N}\mathfrak{N}\delta\beta cl(M)$, we have $(nCy\mathcal{N}\mathfrak{N}\delta\beta cl(M))^c \subseteq M^c$, and hence

$$L \subseteq nCy\mathcal{N}\mathfrak{N}\delta\beta cl(L) \subseteq (nCy\mathcal{N}\mathfrak{N}\delta\beta cl(M))^c \subseteq M^c.$$

Thus L and M are separated $nCy\mathcal{N}\mathfrak{N}\delta\beta os$'s with $A \subseteq M$ and $x_p \in L$. Hence U is $nCy\mathcal{N}\mathfrak{N}\delta\beta Reg$.

The proofs for $nCy\mathcal{N}\mathfrak{N}Reg$, $nCy\mathcal{N}\mathfrak{N}\delta Reg$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{P}Reg$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}Reg$ and $nCy\mathcal{N}\mathfrak{N}\delta\alpha Reg$ spaces are similar.

Definition 3.9 Let $(U_1, \tau_R^{nCy\mathcal{N}}(A_1))$ and $(U_2, \tau_R^{nCy\mathcal{N}}(A_2))$ be two $nCy\mathcal{N}\mathfrak{N}ts$'s. A function $h_{\mathcal{C}}: (U_1, \tau_R^{nCy\mathcal{N}}(A_1)) \to (U_2, \tau_R^{nCy\mathcal{N}}(A_2))$ is said to be n-Cylindrical neutrosophic nano δ (resp. δ pre, δ semi, $\delta\alpha$ and $\delta\beta$) irresolute (briefly, $nCy\mathcal{N}\mathfrak{N}\delta Irr$ (resp. $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{P}Irr$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}Irr$, $nCy\mathcal{N}\mathfrak{N}\delta\alpha Irr$ and $nCy\mathcal{N}\mathfrak{N}\delta\beta Irr$)) if $h_{\mathcal{C}}^{-1}(G)$ is an $nCy\mathcal{N}\mathfrak{N}\delta os$ (resp. $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{P}os$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}os$, $nCy\mathcal{N}\mathfrak{N}\delta\alpha os$ and $nCy\mathcal{N}\mathfrak{N}\delta\beta os$) in U_1 for every $nCy\mathcal{N}\mathfrak{N}\delta os$ (resp. $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{P}os$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}os$, $nCy\mathcal{N}\mathfrak{N}\delta\alpha os$ and $nCy\mathcal{N}\mathfrak{N}\delta\beta os$) G in U_2. Since $h_{\mathcal{C}}^{-1}(T^c) = (h_{\mathcal{C}}^{-1}(T))^c$ for every $nCy\mathcal{N}s$ T in U_2, $h_{\mathcal{C}}$ is $nCy\mathcal{N}\mathfrak{N}\delta\beta Irr$ if and only if $h_{\mathcal{C}}^{-1}(A)$ is an $nCy\mathcal{N}\mathfrak{N}\delta\beta cs$ in U_1 for every $nCy\mathcal{N}\mathfrak{N}\delta\beta cs$ A in U_2; similarly for the other cases.

Theorem 3.4 Let $(U_1, \tau_R^{nCy\mathcal{N}}(F_1))$ and $(U_2, \tau_R^{nCy\mathcal{N}}(F_2))$ be two $nCy\mathcal{N}\mathfrak{N}ts$'s and let $h_{\mathcal{C}}: U_1 \to U_2$ be a bijection. If $h_{\mathcal{C}}$ is $nCy\mathcal{N}\mathfrak{N}\delta\beta Irr$ (resp. $nCy\mathcal{N}\mathfrak{N}Cts$, $nCy\mathcal{N}\mathfrak{N}\delta Irr$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{P}Irr$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}Irr$ and $nCy\mathcal{N}\mathfrak{N}\delta\alpha Irr$) and $nCy\mathcal{N}\mathfrak{N}\delta\beta O$ (resp. $nCy\mathcal{N}\mathfrak{N}O$, $nCy\mathcal{N}\mathfrak{N}\delta O$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{P}O$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}O$ and $nCy\mathcal{N}\mathfrak{N}\delta\alpha O$), and U_1 is $nCy\mathcal{N}\mathfrak{N}\delta\beta Reg$ (resp. $nCy\mathcal{N}\mathfrak{N}Reg$, $nCy\mathcal{N}\mathfrak{N}\delta Reg$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{P}Reg$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}Reg$ and $nCy\mathcal{N}\mathfrak{N}\delta\alpha Reg$), then U_2 is $nCy\mathcal{N}\mathfrak{N}\delta\beta Reg$ (resp. $nCy\mathcal{N}\mathfrak{N}Reg$, $nCy\mathcal{N}\mathfrak{N}\delta Reg$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{P}Reg$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}Reg$ and $nCy\mathcal{N}\mathfrak{N}\delta\alpha Reg$).

Proof. We use the following facts, which follow from Definition 3.5 since $h_{\mathcal{C}}$ is a bijection. For $nCy\mathcal{N}s$'s S, T in U_1 and A in U_2, the value of $h_{\mathcal{C}}(S)$ at $h_{\mathcal{C}}(x)$ equals the value of S at x for every $x \in U_1$; hence $S \subseteq T$ implies $h_{\mathcal{C}}(S) \subseteq h_{\mathcal{C}}(T)$, $h_{\mathcal{C}}(S^c) = (h_{\mathcal{C}}(S))^c$ and $h_{\mathcal{C}}(h_{\mathcal{C}}^{-1}(A)) = A$. Moreover, if x_p is an n-Cylindrical neutrosophic point in U_1, then $h_{\mathcal{C}}(x_p) = (h_{\mathcal{C}}(x))_p$.

Suppose that U_1 is $nCy\mathcal{N}\mathfrak{N}\delta\beta Reg$. Let A be an $nCy\mathcal{N}\mathfrak{N}\delta\beta cs$ in U_2 and let $y_p \in A^c$, where $p = \langle a, b, c\rangle$. Since $h_{\mathcal{C}}$ is $nCy\mathcal{N}\mathfrak{N}\delta\beta Irr$, $h_{\mathcal{C}}^{-1}(A)$ is an $nCy\mathcal{N}\mathfrak{N}\delta\beta cs$ in U_1. Since $h_{\mathcal{C}}$ is a surjection, there exists $x \in U_1$ such that $h_{\mathcal{C}}(x) = y$. Since $y_p \in A^c$, we have $a \le \gamma_A(y)$, $b \le 1 - \beta_A(y)$ and $c \ge \alpha_A(y)$; that is, $a \le \gamma_A(h_{\mathcal{C}}(x))$, $b \le 1 - \beta_A(h_{\mathcal{C}}(x))$ and $c \ge \alpha_A(h_{\mathcal{C}}(x))$. Hence $x_p \in (h_{\mathcal{C}}^{-1}(A))^c$. Since U_1 is $nCy\mathcal{N}\mathfrak{N}\delta\beta Reg$, there exist separated $nCy\mathcal{N}\mathfrak{N}\delta\beta os$'s L and M in U_1 such that $h_{\mathcal{C}}^{-1}(A) \subseteq L$ and $x_p \in M$, where $M \subseteq L^c$. Since $h_{\mathcal{C}}$ is $nCy\mathcal{N}\mathfrak{N}\delta\beta O$, $h_{\mathcal{C}}(L)$ and $h_{\mathcal{C}}(M)$ are $nCy\mathcal{N}\mathfrak{N}\delta\beta os$'s in U_2. Now

$$A = h_{\mathcal{C}}(h_{\mathcal{C}}^{-1}(A)) \subseteq h_{\mathcal{C}}(L) \quad \text{and} \quad y_p = h_{\mathcal{C}}(x_p) \subseteq h_{\mathcal{C}}(M),$$

that is, $y_p \in h_{\mathcal{C}}(M)$. Also, $M \subseteq L^c$ implies $h_{\mathcal{C}}(M) \subseteq h_{\mathcal{C}}(L^c) = (h_{\mathcal{C}}(L))^c$, so $h_{\mathcal{C}}(L)$ and $h_{\mathcal{C}}(M)$ are separated. Hence U_2 is $nCy\mathcal{N}\mathfrak{N}\delta\beta Reg$.

The proofs of the other cases are similar.

Theorem 3.5 Let $(U, \tau_R^{nCy\mathcal{N}}(F))$ be an $nCy\mathcal{N}\mathfrak{N}\delta\beta Reg$ space. Then:

(i) every $nCy\mathcal{N}\mathfrak{N}\delta\beta os$ in U is a union of $nCy\mathcal{N}\mathfrak{N}\delta\beta cs$'s;

(ii) every $nCy\mathcal{N}\mathfrak{N}\delta\beta cs$ in U is an intersection of $nCy\mathcal{N}\mathfrak{N}\delta\beta os$'s.

Proof. (i) Let G be an $nCy\mathcal{N}\mathfrak{N}\delta\beta os$ in U. For each $x \in U$, the n-Cylindrical neutrosophic point $x_{G(x)}$ satisfies $x_{G(x)} \in G$. Since U is $nCy\mathcal{N}\mathfrak{N}\delta\beta Reg$, by Theorem 3.1 ((i) $\Rightarrow$ (ii)) there exists an $nCy\mathcal{N}\mathfrak{N}\delta\beta os$ L_x such that

$$x_{G(x)} \in L_x \subseteq nCy\mathcal{N}\mathfrak{N}\delta\beta cl(L_x) \subseteq G.$$

Put $K_x = nCy\mathcal{N}\mathfrak{N}\delta\beta cl(L_x)$. Then K_x is an $nCy\mathcal{N}\mathfrak{N}\delta\beta cs$ and $x_{G(x)} \subseteq K_x \subseteq G$ for every $x \in U$. Hence, by Definition 3.8(i),

$$G = \bigcup_{x\in U} x_{G(x)} \subseteq \bigcup_{x\in U} K_x \subseteq G,$$

and so $G = \bigcup_{x\in U} K_x$. Thus G is a union of $nCy\mathcal{N}\mathfrak{N}\delta\beta cs$'s.

(ii) Let A be an $nCy\mathcal{N}\mathfrak{N}\delta\beta cs$ in U. Then A^c is an $nCy\mathcal{N}\mathfrak{N}\delta\beta os$, and by (i) there exist $nCy\mathcal{N}\mathfrak{N}\delta\beta cs$'s K_x ($x \in U$) such that $A^c = \bigcup_{x\in U} K_x$. Since the complement $S^c = \{\langle x, \gamma_S(x), 1 - \beta_S(x), \alpha_S(x)\rangle : x \in U\}$ satisfies $(S^c)^c = S$ and $(\bigcup_{x\in U} K_x)^c = \bigcap_{x\in U} K_x^c$ (because $\inf\{1 - \beta_{K_x}\} = 1 - \sup\{\beta_{K_x}\}$), we get

$$A = (A^c)^c = (\bigcup_{x\in U} K_x)^c = \bigcap_{x\in U} K_x^c,$$

where each K_x^c is an $nCy\mathcal{N}\mathfrak{N}\delta\beta os$. Thus A is an intersection of $nCy\mathcal{N}\mathfrak{N}\delta\beta os$'s.

The proofs for $nCy\mathcal{N}\mathfrak{N}Reg$, $nCy\mathcal{N}\mathfrak{N}\delta Reg$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{P}Reg$, $nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}Reg$ and $nCy\mathcal{N}\mathfrak{N}\delta\alpha Reg$ spaces are similar.

Definition 3.10 Let $(U_1, \tau_R^{nCy\mathcal{N}}(A_1))$ and $(U_2, \tau_R^{nCy\mathcal{N}}(A_2))$ be two $nCy\mathcal{N}\mathfrak{N}ts$'s. A function $h_{\mathcal{C}}: (U_1, \tau_R^{nCy\mathcal{N}}(A_1)) \to (U_2, \tau_R^{nCy\mathcal{N}}(A_2))$ is said to be n-Cylindrical neutrosophic nano closed (briefly, $nCy\mathcal{N}\mathfrak{N}C$) if $h_{\mathcal{C}}(K)$ is an $nCy\mathcal{N}\mathfrak{N}cs$ in U_2 for every $nCy\mathcal{N}\mathfrak{N}cs$ K in U_1, where $h_{\mathcal{C}}(K)$ is the image of K in the sense of Definition 3.5

Theorem 3.6 Let $(U_1, \tau_R^{nCy\mathcal{N}}(F_1))$ and $(U_2, \tau_R^{nCy\mathcal{N}}(F_2))$ be two nCy$\mathcal{N}\mathfrak{N}$ts's and let $h_{\mathcal{C}}: U_1 \to U_2$ be an nCy$\mathcal{N}\mathfrak{N}\delta\beta$Cts (resp. nCy$\mathcal{N}\mathfrak{N}Cts, nCy\mathcal{N}\mathfrak{N}\delta$Cts, nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}Cts, nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}$Cts and nCy$\mathcal{N}\mathfrak{N}\delta\alpha$Cts) and nCy$\mathcal{N}\mathfrak{N}$C injection. If U_2 is nCy$\mathcal{N}\mathfrak{N}$Reg and every nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs (resp. nCy$\mathcal{N}\mathfrak{N}cs, nCy\mathcal{N}\mathfrak{N}\delta$cs, nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}cs, nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}$cs and nCy$\mathcal{N}\mathfrak{N}\delta\alpha$cs) in U_1 is an nCy$\mathcal{N}\mathfrak{N}$cs, then U_1 is nCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg (resp. nCy$\mathcal{N}\mathfrak{N}Reg, nCy\mathcal{N}\mathfrak{N}\delta$Reg, nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}Reg, nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}$Reg and nCy$\mathcal{N}\mathfrak{N}\delta\alpha$Reg).

Proof. We use the following facts, which follow from Definition 3.5 For nCy$\mathcal{N}$s's T, T_1, T_2 in U_2, $T_1 \subseteq T_2$ implies $h_{\mathcal{C}}^{-1}(T_1) \subseteq h_{\mathcal{C}}^{-1}(T_2)$ and $h_{\mathcal{C}}^{-1}(T^c) = (h_{\mathcal{C}}^{-1}(T))^c$. Since $h_{\mathcal{C}}$ is an injection, $h_{\mathcal{C}}^{-1}(h_{\mathcal{C}}(x)) = \{x\}$ for every $x \in U_1$, so the value of $h_{\mathcal{C}}(S)$ at $h_{\mathcal{C}}(x)$ equals the value of S at x for every nCy$\mathcal{N}$s S in U_1; consequently $S \subseteq h_{\mathcal{C}}^{-1}(h_{\mathcal{C}}(S))$ and $h_{\mathcal{C}}(x_p) = (h_{\mathcal{C}}(x))_p$.

Let A be an nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs in U_1 and let $x_p \in A^c$, where $p = \langle a, b, c\rangle$. By hypothesis, A is an nCy$\mathcal{N}\mathfrak{N}$cs in U_1. Since $h_{\mathcal{C}}$ is nCy$\mathcal{N}\mathfrak{N}$C, $h_{\mathcal{C}}(A)$ is an nCy$\mathcal{N}\mathfrak{N}$cs in U_2. Put $y = h_{\mathcal{C}}(x)$. Since $x_p \in A^c$, we have $a \le \gamma_A(x) = \gamma_{h_{\mathcal{C}}(A)}(y)$, $b \le 1 - \beta_A(x) = 1 - \beta_{h_{\mathcal{C}}(A)}(y)$ and $c \ge \alpha_A(x) = \alpha_{h_{\mathcal{C}}(A)}(y)$, so $y_p \in (h_{\mathcal{C}}(A))^c$. Since U_2 is nCy$\mathcal{N}\mathfrak{N}$Reg, there exist separated nCy$\mathcal{N}\mathfrak{N}$os's M_1 and M_2 in U_2 such that $y_p \in M_1$, $h_{\mathcal{C}}(A) \subseteq M_2$ and $M_1 \subseteq M_2^c$. Since $h_{\mathcal{C}}$ is nCy$\mathcal{N}\mathfrak{N}\delta\beta$Cts, $h_{\mathcal{C}}^{-1}(M_1)$ and $h_{\mathcal{C}}^{-1}(M_2)$ are nCy$\mathcal{N}\mathfrak{N}\delta\beta$os's in U_1. Now the value of $h_{\mathcal{C}}^{-1}(M_1)$ at x is the value of M_1 at y, and $y_p \in M_1$; hence $x_p \in h_{\mathcal{C}}^{-1}(M_1)$. Also,

$A \subseteq h_{\mathcal{C}}^{-1}(h_{\mathcal{C}}(A)) \subseteq h_{\mathcal{C}}^{-1}(M_2)$ and $h_{\mathcal{C}}^{-1}(M_1) \subseteq h_{\mathcal{C}}^{-1}(M_2^c) = (h_{\mathcal{C}}^{-1}(M_2))^c$.

Thus $h_{\mathcal{C}}^{-1}(M_2)$ and $h_{\mathcal{C}}^{-1}(M_1)$ are separated nCy$\mathcal{N}\mathfrak{N}\delta\beta$os's with $A \subseteq h_{\mathcal{C}}^{-1}(M_2)$ and $x_p \in h_{\mathcal{C}}^{-1}(M_1)$. Hence U_1 is nCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg.

The proofs of the other cases are similar.

Theorem 3.7 Let $(U_1, \tau_R^{nCy\mathcal{N}}(F_1))$ and $(U_2, \tau_R^{nCy\mathcal{N}}(F_2))$ be two nCy$\mathcal{N}\mathfrak{N}$ts's and let $h_{\mathcal{C}}: U_1 \to U_2$ be an nCy$\mathcal{N}\mathfrak{N}$Cts and nCy$\mathcal{N}\mathfrak{N}\delta\beta$O (resp. nCy$\mathcal{N}\mathfrak{N}O, nCy\mathcal{N}\mathfrak{N}\delta$O, nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}O, nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}$O and nCy$\mathcal{N}\mathfrak{N}\delta\alpha$O) bijection. If U_1 is nCy$\mathcal{N}\mathfrak{N}$Reg and every nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs (resp. nCy$\mathcal{N}\mathfrak{N}cs, nCy\mathcal{N}\mathfrak{N}\delta$cs, nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}cs, nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}$cs and nCy$\mathcal{N}\mathfrak{N}\delta\alpha$cs) in U_2 is an nCy$\mathcal{N}\mathfrak{N}$cs, then U_2 is nCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg (resp. nCy$\mathcal{N}\mathfrak{N}Reg, nCy\mathcal{N}\mathfrak{N}\delta$Reg, nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}Reg, nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}$Reg and nCy$\mathcal{N}\mathfrak{N}\delta\alpha$Reg).

Proof. As in the proof of Theorem 3.4, since $h_{\mathcal{C}}$ is a bijection, for nCy$\mathcal{N}$s's S, T in U_1 and B in U_2 we have: $S \subseteq T$ implies $h_{\mathcal{C}}(S) \subseteq h_{\mathcal{C}}(T)$, $h_{\mathcal{C}}(S^c) = (h_{\mathcal{C}}(S))^c$, $h_{\mathcal{C}}(h_{\mathcal{C}}^{-1}(B)) = B$ and $h_{\mathcal{C}}(x_p) = (h_{\mathcal{C}}(x))_p$ for every n-Cylindrical neutrosophic point x_p in U_1.

Let B be an nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs in U_2 and let $y_p \in B^c$, where $p = \langle a, b, c\rangle$. By hypothesis, B is an nCy$\mathcal{N}\mathfrak{N}$cs in U_2. Since $h_{\mathcal{C}}$ is nCy$\mathcal{N}\mathfrak{N}$Cts, $h_{\mathcal{C}}^{-1}(B)$ is an nCy$\mathcal{N}\mathfrak{N}$cs in U_1. Since $h_{\mathcal{C}}$ is a surjection, there exists $x \in U_1$ such that $h_{\mathcal{C}}(x) = y$. Since $y_p \in B^c$, we have $a \le \gamma_B(h_{\mathcal{C}}(x))$, $b \le 1 - \beta_B(h_{\mathcal{C}}(x))$ and $c \ge \alpha_B(h_{\mathcal{C}}(x))$, that is, $x_p \in (h_{\mathcal{C}}^{-1}(B))^c$. Since U_1 is nCy$\mathcal{N}\mathfrak{N}$Reg, there exist separated nCy$\mathcal{N}\mathfrak{N}$os's L_1 and L_2 in U_1 such that $x_p \in L_1$, $h_{\mathcal{C}}^{-1}(B) \subseteq L_2$ and $L_1 \subseteq L_2^c$. Since $h_{\mathcal{C}}$ is nCy$\mathcal{N}\mathfrak{N}\delta\beta$O, $h_{\mathcal{C}}(L_1)$ and $h_{\mathcal{C}}(L_2)$ are nCy$\mathcal{N}\mathfrak{N}\delta\beta$os's in U_2. Now

$y_p = h_{\mathcal{C}}(x_p) \subseteq h_{\mathcal{C}}(L_1)$ and $B = h_{\mathcal{C}}(h_{\mathcal{C}}^{-1}(B)) \subseteq h_{\mathcal{C}}(L_2)$,

that is, $y_p \in h_{\mathcal{C}}(L_1)$ and $B \subseteq h_{\mathcal{C}}(L_2)$. Also, $L_1 \subseteq L_2^c$ implies $h_{\mathcal{C}}(L_1) \subseteq h_{\mathcal{C}}(L_2^c) = (h_{\mathcal{C}}(L_2))^c$, so $h_{\mathcal{C}}(L_2)$ and $h_{\mathcal{C}}(L_1)$ are separated. Hence U_2 is nCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg.

The proofs of the other cases are similar.

Remark 3.1 Theorems 3.1, 3.2, 3.3, 3.4, 3.5, 3.6 and 3.7 have been stated and proved for nCy$\mathcal{N}\mathfrak{N}\delta\beta$os's. They also hold for nCy$\mathcal{N}\mathfrak{N}$os's, nCy$\mathcal{N}\mathfrak{N}\delta$os's, nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$os's, nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{S}$os's and nCy$\mathcal{N}\mathfrak{N}\delta\alpha$os's, with nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs's, nCy$\mathcal{N}\mathfrak{N}\delta\beta$cl and nCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg replaced by the corresponding closed sets, closure operators and regular spaces (for nCy$\mathcal{N}\mathfrak{N}$os's, this is Corollary 3.1 in the case of Theorem 3.1). Indeed, the proofs use only the following properties of the class of nCy$\mathcal{N}\mathfrak{N}\delta\beta$os's, which each of these classes also has:

(i) an nCy$\mathcal{N}$s S belongs to the class if and only if S^c is a closed set of the corresponding type;

(ii) any union of members of the class belongs to the class, so the corresponding closure of S is the smallest closed set of that type containing S.

4 Strongly n-Cylindrical neutrosophic nano (resp. δ, δ pre, δ semi, δα and δβ) regular spaces

In this section, we introduce strongly n-Cylindrical neutrosophic nano (resp. δ, δ pre, δ semi, δα and δβ) regular spaces and study their properties.

Definition 4.1 An nCy$\mathcal{N}\mathfrak{N}$ts $(U, \tau_R^{nCy\mathcal{N}}(F))$ is said to be strongly n-Cylindrical neutrosophic nano (resp. δ, δ pre, δ semi, δα and δβ) regular (briefly, StnCy$\mathcal{N}\mathfrak{N}$Reg (resp. StnCy$\mathcal{N}\mathfrak{N}\delta$Reg, StnCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$Reg, StnCy$\mathcal{N}\mathfrak{N}\delta\mathcal{S}$Reg, StnCy$\mathcal{N}\mathfrak{N}\delta\alpha$Reg and StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg)) if for every nCy$\mathcal{N}\mathfrak{N}$cs (resp. nCy$\mathcal{N}\mathfrak{N}\delta$cs, nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}cs, nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}cs, nCy\mathcal{N}\mathfrak{N}\delta\alpha$cs and nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs) A in U and every n-Cylindrical neutrosophic point $x_p \in A^c$, there exist separated nCy$\mathcal{N}\mathfrak{N}$os's L and M in U such that $A \subseteq L$ and $x_p \in M$.

Proposition 4.1 Let $(U, \tau_R^{nCy\mathcal{N}}(F))$ be an nCy$\mathcal{N}\mathfrak{N}$ts.

(i) If U is StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg (resp. StnCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$Reg), then U is nCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg (resp. nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$Reg).

(ii) If U is StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg (resp. StnCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$Reg), then U is nCy$\mathcal{N}\mathfrak{N}$Reg.

Proof. First we show that every nCy$\mathcal{N}\mathfrak{N}$os is an nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$os and an nCy$\mathcal{N}\mathfrak{N}\delta\beta$os. Let S be an nCy$\mathcal{N}$s in U. Every nCy$\mathcal{N}\mathfrak{N}$rc set is an nCy$\mathcal{N}\mathfrak{N}$cs, so every nCy$\mathcal{N}\mathfrak{N}$rc set containing S contains nCy$\mathcal{N}\mathfrak{N}$cl(S); hence nCy$\mathcal{N}\mathfrak{N}$cl(S) ⊆ nCy$\mathcal{N}\mathfrak{N}\delta$cl(S). Now let G be an nCy$\mathcal{N}\mathfrak{N}$os. Then

$$G = nCy\mathcal{N}\mathfrak{N}int(G) \subseteq nCy\mathcal{N}\mathfrak{N}int(nCy\mathcal{N}\mathfrak{N}\delta cl(G)) \subseteq nCy\mathcal{N}\mathfrak{N}cl(nCy\mathcal{N}\mathfrak{N}int(nCy\mathcal{N}\mathfrak{N}\delta cl(G))),$$

so G is an nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$os and an nCy$\mathcal{N}\mathfrak{N}\delta\beta$os. Taking complements, every nCy$\mathcal{N}\mathfrak{N}$cs is an nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$cs and an nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs.

(i) Suppose that U is StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg. Let A be an nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs in U and let $x_p \in A^c$. Since U is StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg, there exist separated nCy$\mathcal{N}\mathfrak{N}$os's L and M such that A ⊆ L and $x_p \in M$. Since every nCy$\mathcal{N}\mathfrak{N}$os is an nCy$\mathcal{N}\mathfrak{N}\delta\beta$os, L and M are separated nCy$\mathcal{N}\mathfrak{N}\delta\beta$os's with A ⊆ L and $x_p \in M$. Hence U is nCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg.

(ii) Suppose that U is StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg. Let A be an nCy$\mathcal{N}\mathfrak{N}$cs in U and let $x_p \in A^c$. Since every nCy$\mathcal{N}\mathfrak{N}$cs is an nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs, A is an nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs. Since U is StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg, there exist separated nCy$\mathcal{N}\mathfrak{N}$os's L and M such that A ⊆ L and $x_p \in M$. Hence U is nCy$\mathcal{N}\mathfrak{N}$Reg.

The proofs for the StnCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$Reg case are similar.

Definition 4.2 An nCy$\mathcal{N}\mathfrak{N}$ts $(U, \tau_R^{nCy\mathcal{N}}(F))$ is said to be strongly * n-Cylindrical neutrosophic nano (resp. δ, δ pre, δ semi, δα and δβ) regular (briefly, St*nCy$\mathcal{N}\mathfrak{N}$Reg (resp. St*nCy$\mathcal{N}\mathfrak{N}\delta$Reg, St*nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$Reg, St*nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{S}$Reg, St*nCy$\mathcal{N}\mathfrak{N}\delta\alpha$Reg and St*nCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg)) if for every nCy$\mathcal{N}\mathfrak{N}$cs A in U and every n-Cylindrical neutrosophic point $x_p \in A^c$, there exist separated nCy$\mathcal{N}\mathfrak{N}$os's (resp. nCy$\mathcal{N}\mathfrak{N}\delta$os's, nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$os's, nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{S}$os's, nCy$\mathcal{N}\mathfrak{N}\delta\alpha$os's and nCy$\mathcal{N}\mathfrak{N}\delta\beta$os's) L and M in U such that A ⊆ L and $x_p \in M$.

Proposition 4.2 Let $(U, \tau_R^{nCy\mathcal{N}}(F))$ be an nCy$\mathcal{N}\mathfrak{N}$ts.

(i) If U is nCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg (resp. nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$Reg), then U is St*nCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg (resp. St*nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$Reg).

(ii) If U is nCy$\mathcal{N}\mathfrak{N}$Reg, then U is St*nCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg and St*nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$Reg.

Proof. By the proof of Proposition 4.1, every nCy$\mathcal{N}\mathfrak{N}$os is an nCy$\mathcal{N}\mathfrak{N}\delta\beta$os and every nCy$\mathcal{N}\mathfrak{N}$cs is an nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs; the same holds for nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$os's and nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$cs's.

(i) Suppose that U is nCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg. Let A be an nCy$\mathcal{N}\mathfrak{N}$cs in U and let $x_p \in A^c$. Since every nCy$\mathcal{N}\mathfrak{N}$cs is an nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs, A is an nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs. Since U is nCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg, there exist separated nCy$\mathcal{N}\mathfrak{N}\delta\beta$os's L and M such that A ⊆ L and $x_p \in M$. Hence U is St*nCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg.

(ii) Suppose that U is nCy$\mathcal{N}\mathfrak{N}$Reg. Let A be an nCy$\mathcal{N}\mathfrak{N}$cs in U and let $x_p \in A^c$. Since U is nCy$\mathcal{N}\mathfrak{N}$Reg, there exist separated nCy$\mathcal{N}\mathfrak{N}$os's L and M such that A ⊆ L and $x_p \in M$. Since every nCy$\mathcal{N}\mathfrak{N}$os is an nCy$\mathcal{N}\mathfrak{N}\delta\beta$os, L and M are separated nCy$\mathcal{N}\mathfrak{N}\delta\beta$os's with A ⊆ L and $x_p \in M$. Hence U is St*nCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg.

The proofs for the nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$ case are similar.

The following examples show that the converses of Propositions 4.1 and 4.2 (ii) need not hold. In both examples n = 3

Example 4.1 Let U = {a}, let R be the identity relation on U and let $F = 0_{nCy\mathcal{N}}$. Then $\underline{nCy\mathcal{N}}(F) = \overline{nCy\mathcal{N}}(F) = B_{nCy\mathcal{N}}(F) = 0_{nCy\mathcal{N}}$, so $\tau_R^{nCy\mathcal{N}}(F) = \{0_{nCy\mathcal{N}}, 1_{nCy\mathcal{N}}\}$, and the nCy$\mathcal{N}\mathfrak{N}$cs's are $0_{nCy\mathcal{N}}$ and $1_{nCy\mathcal{N}}$. Hence the only nCy$\mathcal{N}\mathfrak{N}$rc sets are $0_{nCy\mathcal{N}}$ and $1_{nCy\mathcal{N}}$, and $nCy\mathcal{N}\mathfrak{N}\delta cl(S) = 1_{nCy\mathcal{N}}$ for every nCy$\mathcal{N}$s $S \neq 0_{nCy\mathcal{N}}$. Consequently,

$$S \subseteq 1_{nCy\mathcal{N}} = nCy\mathcal{N}\mathfrak{N}int(nCy\mathcal{N}\mathfrak{N}\delta cl(S)) = nCy\mathcal{N}\mathfrak{N}cl(nCy\mathcal{N}\mathfrak{N}int(nCy\mathcal{N}\mathfrak{N}\delta cl(S)))$$

for every $S \neq 0_{nCy\mathcal{N}}$, that is, every nCy$\mathcal{N}$s in U is an nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$os and an nCy$\mathcal{N}\mathfrak{N}\delta\beta$os, and hence also an nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$cs and an nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs.

(i) U is nCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg and nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$Reg: if A is any nCy$\mathcal{N}$s and $x_p \in A^c$, then L = A and $M = x_p$ are nCy$\mathcal{N}\mathfrak{N}\delta\beta$os's (and nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$os's) with A ⊆ L, $x_p \in M$ and $M \subseteq A^c = L^c$.

(ii) U is nCy$\mathcal{N}\mathfrak{N}$Reg: for $A = 0_{nCy\mathcal{N}}$, the sets $L = 0_{nCy\mathcal{N}}$ and $M = 1_{nCy\mathcal{N}}$ are separated, since $L \subseteq 0_{nCy\mathcal{N}} = M^c$; for $A = 1_{nCy\mathcal{N}}$, the only point in $A^c = 0_{nCy\mathcal{N}}$ is $a_{\langle 0,0,1\rangle} = 0_{nCy\mathcal{N}}$, and $L = 1_{nCy\mathcal{N}}$, $M = 0_{nCy\mathcal{N}}$ are separated.

(iii) U is not StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg: let $A = \{\langle a, 0.5, 0.5, 0.5\rangle\}$, which is an nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs, and $p = \langle 0.4, 0.4, 0.6\rangle$. Since $A^c = \{\langle a, 0.5, 0.5, 0.5\rangle\}$, we have $a_p \in A^c$. The only nCy$\mathcal{N}\mathfrak{N}$os containing A is $L = 1_{nCy\mathcal{N}}$, and then $M \subseteq L^c = 0_{nCy\mathcal{N}}$ cannot contain a_p. Similarly, U is not StnCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$Reg.

Thus U is nCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg (resp. nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$Reg) but not StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg (resp. StnCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$Reg), and U is nCy$\mathcal{N}\mathfrak{N}$Reg but not StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg. Hence the converses of Proposition 4.1(i) and (ii) need not hold.

Example 4.2 Let U = {a, b}, let R be the identity relation on U (so that $[a]_R = \{a\}$ and $[b]_R = \{b\}$) and let

$$F = \{\langle a, 1,1,0\rangle, \langle b, 0,0,1\rangle\}.$$

Then $\underline{\mathrm{nCy}\mathcal{N}}(F) = \overline{\mathrm{nCy}\mathcal{N}}(F) = F$ and $F^c = \{\langle a,0,0,1\rangle, \langle b,1,1,0\rangle\}$, so $B_{\mathrm{nCy}\mathcal{N}}(F) = F \cap F^c = 0_{\mathrm{nCy}\mathcal{N}}$. Hence $\tau_R^{\mathrm{nCy}\mathcal{N}}(F) = \{0_{\mathrm{nCy}\mathcal{N}}, 1_{\mathrm{nCy}\mathcal{N}}, F\}$, and the $\mathrm{nCy}\mathcal{N}\mathfrak{N}$cs's are $1_{\mathrm{nCy}\mathcal{N}}$, F^c and $0_{\mathrm{nCy}\mathcal{N}}$.

Since $F \not\supseteq F^c$, we have $\mathrm{nCy}\mathcal{N}\mathfrak{N}\mathrm{cl}(F) = 1_{\mathrm{nCy}\mathcal{N}}$ and $\mathrm{nCy}\mathcal{N}\mathfrak{N}\mathrm{int}(\mathrm{nCy}\mathcal{N}\mathfrak{N}\mathrm{cl}(F)) = 1_{\mathrm{nCy}\mathcal{N}} \neq F$, so F is not an $\mathrm{nCy}\mathcal{N}\mathfrak{N}$ro set. Since $F \not\supseteq F^c$, also $\mathrm{nCy}\mathcal{N}\mathfrak{N}\mathrm{int}(F^c) = 0_{\mathrm{nCy}\mathcal{N}}$ and $\mathrm{nCy}\mathcal{N}\mathfrak{N}\mathrm{cl}(\mathrm{nCy}\mathcal{N}\mathfrak{N}\mathrm{int}(F^c)) = 0_{\mathrm{nCy}\mathcal{N}} \neq F^c$, so F^c is not an $\mathrm{nCy}\mathcal{N}\mathfrak{N}$rc set. Hence the only $\mathrm{nCy}\mathcal{N}\mathfrak{N}$ro (resp. $\mathrm{nCy}\mathcal{N}\mathfrak{N}$rc) sets are $0_{\mathrm{nCy}\mathcal{N}}$ and $1_{\mathrm{nCy}\mathcal{N}}$, so $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\mathrm{int}(S) = 0_{\mathrm{nCy}\mathcal{N}}$ for $S \neq 1_{\mathrm{nCy}\mathcal{N}}$ and $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\mathrm{cl}(S) = 1_{\mathrm{nCy}\mathcal{N}}$ for $S \neq 0_{\mathrm{nCy}\mathcal{N}}$.

(i) F is an $\mathrm{nCy}\mathcal{N}\mathfrak{N}$os, but it is not an $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta$os, an $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\mathcal{S}$os or an $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\alpha$os, since $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\mathrm{int}(F) = 0_{\mathrm{nCy}\mathcal{N}} \neq F$, $\mathrm{nCy}\mathcal{N}\mathfrak{N}\mathrm{cl}(\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\mathrm{int}(F)) = 0_{\mathrm{nCy}\mathcal{N}}$ and $\mathrm{nCy}\mathcal{N}\mathfrak{N}\mathrm{int}(\mathrm{nCy}\mathcal{N}\mathfrak{N}\mathrm{cl}(\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\mathrm{int}(F))) = 0_{\mathrm{nCy}\mathcal{N}}$.

(ii) As in Example 4.1, every $\mathrm{nCy}\mathcal{N}$s in U is an $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\beta$os and an $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\mathcal{P}$os, so U is $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\beta$Reg, $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\mathcal{P}$Reg, $\mathrm{St}^*\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\beta$Reg and $\mathrm{St}^*\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\mathcal{P}$Reg (take $L = A$ and $M = x_p$).

(iii) U is not $\mathrm{nCy}\mathcal{N}\mathfrak{N}$Reg: let $A = F^c$ and $p = \langle 0.5,0.5,0.5\rangle$. Then $a_p \in F = A^c$. Since the value of A at b is $\langle 1,1,0\rangle$, the only $\mathrm{nCy}\mathcal{N}\mathfrak{N}$os containing A is $L = 1_{\mathrm{nCy}\mathcal{N}}$, and then $M \subseteq L^c = 0_{\mathrm{nCy}\mathcal{N}}$ cannot contain a_p. For the same reason, U is not $\mathrm{StnCy}\mathcal{N}\mathfrak{N}\delta\beta$Reg.

Thus U is $\mathrm{St}^*\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\beta$Reg and $\mathrm{St}^*\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\mathcal{P}$Reg but not $\mathrm{nCy}\mathcal{N}\mathfrak{N}$Reg, so the converse of Proposition 4.2(ii) need not hold, and U is $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\beta$Reg but not $\mathrm{nCy}\mathcal{N}\mathfrak{N}$Reg.

In Examples 4.1 and 4.2, every $\mathrm{nCy}\mathcal{N}$s is an $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\beta$os, so $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\beta$-regularity holds for trivial reasons. The next example gives a space in which some $\mathrm{nCy}\mathcal{N}$s's are not $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\beta$os's, but which is still $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\beta$Reg.

Example 4.3 Let $U = \{a, b\}$, let $R = U \times U$ (so that U is a single equivalence class) and let

$$F = \{\langle a,0,1,1\rangle, \langle b,0.5,1,0\rangle\}.$$

For $v \in [0,1]^3$ with admissible values, we write $\hat{v}$ for the constant $\mathrm{nCy}\mathcal{N}$s in U with value v at a and at b. By Definition 2.14,

$$\underline{\mathrm{nCy}\mathcal{N}}(F) = G_1 = \langle\widehat{0,1,1}\rangle, \quad \overline{\mathrm{nCy}\mathcal{N}}(F) = G_3 = \langle\widehat{0.5,1,0}\rangle,$$
$$B_{\mathrm{nCy}\mathcal{N}}(F) = G_3 \cap G_1^c = G_2 = \langle\widehat{0.5,0,0}\rangle.$$

Since $G_1 \cup G_2 = G_3$, $G_1 \cap G_2 = 0_{\mathrm{nCy}\mathcal{N}}$ and $G_1, G_2 \subseteq G_3$, the family $\tau_R^{\mathrm{nCy}\mathcal{N}}(F) = \{0_{\mathrm{nCy}\mathcal{N}}, 1_{\mathrm{nCy}\mathcal{N}}, G_1, G_2, G_3\}$ is an $\mathrm{nCy}\mathcal{N}$t. Its $\mathrm{nCy}\mathcal{N}\mathfrak{N}$cs's are $0_{\mathrm{nCy}\mathcal{N}}$, $1_{\mathrm{nCy}\mathcal{N}}$,

$$G_1^c = \langle\widehat{1,0,0}\rangle, \quad G_2^c = \langle\widehat{0,1,0.5}\rangle \quad \text{and} \quad G_3^c = \langle\widehat{0,0,0.5}\rangle.$$

A direct check gives $\mathrm{nCy}\mathcal{N}\mathfrak{N}\mathrm{int}(G_1^c) = G_2$, $\mathrm{nCy}\mathcal{N}\mathfrak{N}\mathrm{cl}(G_2) = G_1^c$, $\mathrm{nCy}\mathcal{N}\mathfrak{N}\mathrm{int}(G_2^c) = G_1$, $\mathrm{nCy}\mathcal{N}\mathfrak{N}\mathrm{cl}(G_1) = G_2^c$ and $\mathrm{nCy}\mathcal{N}\mathfrak{N}\mathrm{int}(G_3^c) = 0_{\mathrm{nCy}\mathcal{N}}$. Hence the $\mathrm{nCy}\mathcal{N}\mathfrak{N}$rc sets are $0_{\mathrm{nCy}\mathcal{N}}$, $1_{\mathrm{nCy}\mathcal{N}}$, G_1^c and G_2^c, and $G_3^c = G_1^c \cap G_2^c$ is an intersection of $\mathrm{nCy}\mathcal{N}\mathfrak{N}$rc sets which is not an $\mathrm{nCy}\mathcal{N}\mathfrak{N}$rc set.

For an $\mathrm{nCy}\mathcal{N}$s S, $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\mathrm{cl}(S)$ is one of $0_{\mathrm{nCy}\mathcal{N}}$, G_3^c, G_1^c, G_2^c and $1_{\mathrm{nCy}\mathcal{N}}$, and $\mathrm{nCy}\mathcal{N}\mathfrak{N}\mathrm{cl}(\mathrm{nCy}\mathcal{N}\mathfrak{N}\mathrm{int}(K)) = K$ for $K = 0_{\mathrm{nCy}\mathcal{N}}, G_1^c, G_2^c, 1_{\mathrm{nCy}\mathcal{N}}$, while $\mathrm{nCy}\mathcal{N}\mathfrak{N}\mathrm{cl}(\mathrm{nCy}\mathcal{N}\mathfrak{N}\mathrm{int}(G_3^c)) = 0_{\mathrm{nCy}\mathcal{N}}$. Therefore S is an $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\beta$os if and only if either $S = 0_{\mathrm{nCy}\mathcal{N}}$ or $S \not\supseteq G_3^c$,

and, taking complements, A is an $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\beta$cs if and only if either $A = 1_{\mathrm{nCy}\mathcal{N}}$ or $G_3 \not\supseteq A$. In particular, G_3^c is not an $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\beta$os.

U is $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\beta$Reg. Let A be an $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\beta$cs and let $x_p \in A^c$, where $p = \langle a_0, b_0, c_0\rangle$; let z be the other point of U. Note that $x_p \in A^c$ means $A \subseteq (x_p)^c$.

(i) If x_p is an $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\beta$os, take $L = (x_p)^c$ and $M = x_p$. Since L has the value $\langle 1,1,0\rangle$ at z, $L \not\supseteq G_3^c$, so L is an $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\beta$os; moreover $A \subseteq L$, $x_p \in M$ and $M = L^c$.

(ii) Otherwise $x_p \neq 0_{\mathrm{nCy}\mathcal{N}}$ and $x_p \subseteq G_3^c$, that is, $p = \langle 0,0,c_0\rangle$ with $\frac{1}{2} \leq c_0 <$ (i) If $A(z) \not\supseteq \langle 0.5,1,0\rangle$, then $A^c(z) \not\supseteq \langle 0,0,0.5\rangle$; take L with values $\langle c_0, 1,0\rangle$ at x and A(z) at z, and M with values p at x and $A^c(z)$ at z. Then $L \not\supseteq G_3^c$ and $M \not\supseteq G_3^c$, so both are $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\beta$os's, $A \subseteq L$, $x_p \in M$ and $M = L^c$.

(iii) Finally, let $p = \langle 0,0,c_0\rangle$ with $\frac{1}{2} \leq c_0 < 1$ and $A(z) \supseteq \langle 0.5,1,0\rangle$. Since $\alpha_A(x) \leq c_0 < 1$, $A \neq 1_{\mathrm{nCy}\mathcal{N}}$, so $G_3 \not\supseteq A$ and therefore $A(x) \not\supseteq \langle 0.5,1,0\rangle$. Let L have the values A(x) at x and $\langle 1,1,0\rangle$ at z, so that L is an $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\beta$os containing A, and let M have the value $\langle 0,0,1\rangle$ at z and the following value m at x: $m = \langle 0,0,c'\rangle$ with $\alpha_A(x) \leq c' < \frac{1}{2}$ if $\alpha_A(x) < \frac{1}{2}$; $m = \langle 0,1-\beta_A(x), c_0\rangle$ if $\beta_A(x) < 1$; and $m = \langle \varepsilon, 0, c_0\rangle$ with $0 < \varepsilon \leq \min\{\gamma_A(x), (1-c_0^n)^{1/n}\}$ if $\gamma_A(x) > 0$. In each case $m \supseteq p$, $m \not\supseteq \langle 0,0,0.5\rangle$ and $A(x) \subseteq m^c$, so M is an $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\beta$os with $x_p \in M$ and $L \subseteq M^c$.

Hence U is $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\beta$Reg. On the other hand, U is not $\mathrm{nCy}\mathcal{N}\mathfrak{N}$Reg: the only $\mathrm{nCy}\mathcal{N}\mathfrak{N}$os containing $G_1^c = \langle\widehat{1,0,0}\rangle$ is $1_{\mathrm{nCy}\mathcal{N}}$, while the n-Cylindrical neutrosophic point $a_{\langle 0,1,1\rangle} \neq 0_{\mathrm{nCy}\mathcal{N}}$ belongs to $G_1 = (G_1^c)^c$.

The following result shows that strong $\delta\beta$-regularity is a very restrictive condition.

Proposition 4.3 Let $(U, \tau_R^{\mathrm{nCy}\mathcal{N}}(F))$ be an $\mathrm{nCy}\mathcal{N}\mathfrak{N}$ts, let $x \in U$, and suppose that, for infinitely many $t \in (0, 2^{-1/n}]$, the n-Cylindrical neutrosophic point $x_{\langle t,1/2,t\rangle}$ is an $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\beta$cs. Then U is not

StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg. In particular, if every nCy$\mathcal{N}$s in U is an nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs and $U \neq \emptyset$, then U is not StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg.

Proof. Suppose that U is StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg. Let t be as in the hypothesis and put $A_t = x_{\langle t,1/2,t\rangle}$; note that $t^n + t^n \leq 1$. The value of A_t^c at x is $\langle t, 1/2, t\rangle$ and its value at every other point is $\langle 1,1,0\rangle$, so $x_{\langle t,1/2,t\rangle} \in A_t^c$. Hence there exist separated nCy$\mathcal{N}\mathfrak{N}$os's L_t and M_t such that $A_t \subseteq L_t$ and $x_{\langle t,1/2,t\rangle} \in M_t$. Then $\alpha_{L_t}(x) \geq t$ and $\gamma_{M_t}(x) \leq t$, and $L_t \subseteq M_t^c$ gives $\alpha_{L_t}(x) \leq \gamma_{M_t}(x)$. Thus $\alpha_{L_t}(x) = t$. Hence the sets L_t are pairwise distinct for the infinitely many values of t, which contradicts the fact that $\tau_R^{nCy\mathcal{N}}(F)$ has at most five members.

Remark 4.1 (i) In Examples 4.1, 4.2 and 4.3, the points $x_{\langle t,1/2,t\rangle}$ are nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs's (in Example 4.3, because $G_3 \not\supseteq x_{\langle t,1/2,t\rangle}$), so by Proposition 4.3 none of these spaces is StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg. We do not know an example of an StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg space. Consequently, Theorems (iv)1–(iv)4 and Corollary 4.1 are mainly of interest for StnCy$\mathcal{N}\mathfrak{N}$Reg spaces, which coincide with nCy$\mathcal{N}\mathfrak{N}$Reg spaces (Example 4.1 is such a space), and they describe what must hold in any StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg space, if one exists.

(ii) We do not know whether the converse of Proposition 4.2(i) holds, that is, whether every St*nCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg space is nCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg.

Theorem 4.1 In an nCy$\mathcal{N}\mathfrak{N}$ts $(U, \tau_R^{nCy\mathcal{N}}(F))$, the following statements are equivalent:

(i) U is StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg;

(ii) for every nCy$\mathcal{N}\mathfrak{N}\delta\beta$os G in U and every n-Cylindrical neutrosophic point $x_p \in G$, there exists an nCy$\mathcal{N}\mathfrak{N}$os L such that $x_p \in L \subseteq nCy\mathcal{N}\mathfrak{N}cl(L) \subseteq G$;

(iii) for every nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs B in U and every n-Cylindrical neutrosophic point $x_p \in B^c$, there exist an nCy$\mathcal{N}\mathfrak{N}$cs N and an nCy$\mathcal{N}\mathfrak{N}$os V such that $B \subseteq V \subseteq N$ and $x_p \in N^c$.

Moreover, if U is StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg, then: [(a)]

(i) for every nCy$\mathcal{N}$s A and every nCy$\mathcal{N}\mathfrak{N}\delta\beta$os B in U with $A \cap B \neq 0_{nCy\mathcal{N}}$, there exists an nCy$\mathcal{N}\mathfrak{N}$os L such that $A \cap L \neq 0_{nCy\mathcal{N}}$ and $nCy\mathcal{N}\mathfrak{N}cl(L) \subseteq B$;

(ii) for every nCy$\mathcal{N}$s $A \neq 0_{nCy\mathcal{N}}$ and every nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs B in U with $A \cap B = 0_{nCy\mathcal{N}}$, there exist separated nCy$\mathcal{N}\mathfrak{N}$os's L and M such that $A \cap L \neq 0_{nCy\mathcal{N}}$ and $B \subseteq M$.

Proof. Recall that nCy$\mathcal{N}\mathfrak{N}$cl(S) is the smallest nCy$\mathcal{N}\mathfrak{N}$cs containing S, that an nCy$\mathcal{N}$s is an nCy$\mathcal{N}\mathfrak{N}$os if and only if its complement is an nCy$\mathcal{N}\mathfrak{N}$cs, and that $L \cap M = 0_{nCy\mathcal{N}}$ implies $M \subseteq L^c$ (see the proof of Theorem 3.1).

(i) ⇒ (ii): Suppose that U is StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg. Let G be an nCy$\mathcal{N}\mathfrak{N}\delta\beta$os in U and let $x_p \in G$. Then G^c is an nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs and $x_p \in (G^c)^c$. Since U is StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg, there exist separated nCy$\mathcal{N}\mathfrak{N}$os's M and L such that $G^c \subseteq M$ and $x_p \in L$. Since $L \subseteq M^c$ and M^c is an nCy$\mathcal{N}\mathfrak{N}$cs, we get $nCy\mathcal{N}\mathfrak{N}cl(L) \subseteq M^c$. Also, $G^c \subseteq M$ implies $M^c \subseteq G$. Hence $x_p \in L \subseteq nCy\mathcal{N}\mathfrak{N}cl(L) \subseteq G$.

(ii) ⇒ (iii): Let B be an nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs in U and let $x_p \in B^c$. Since B^c is an nCy$\mathcal{N}\mathfrak{N}\delta\beta$os, by (ii) there exists an nCy$\mathcal{N}\mathfrak{N}$os L such that $x_p \in L \subseteq nCy\mathcal{N}\mathfrak{N}cl(L) \subseteq B^c$. Put $N = L^c$ and $V = (nCy\mathcal{N}\mathfrak{N}cl(L))^c$. Then N is an nCy$\mathcal{N}\mathfrak{N}$cs, V is an nCy$\mathcal{N}\mathfrak{N}$os, and taking complements in $L \subseteq nCy\mathcal{N}\mathfrak{N}cl(L) \subseteq B^c$ gives $B \subseteq V \subseteq N$. Moreover, $x_p \in L = N^c$.

(iii) ⇒ (i): Let B be an nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs in U and let $x_p \in B^c$. By (iii), there exist an nCy$\mathcal{N}\mathfrak{N}$cs N and an nCy$\mathcal{N}\mathfrak{N}$os V such that $B \subseteq V \subseteq N$ and $x_p \in N^c$. Put $L = V$ and $M = N^c$. Then L and M are nCy$\mathcal{N}\mathfrak{N}$os's, $B \subseteq L$ and $x_p \in M$. Since $V \subseteq N$, we have $L \subseteq N = M^c$, so L and M are separated. Hence U is StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg.

(a) Let A be an nCy$\mathcal{N}$s and let B be an nCy$\mathcal{N}\mathfrak{N}\delta\beta$os with $A \cap B \neq 0_{nCy\mathcal{N}}$. Then there exists $x \in U$ such that $(A \cap B)(x) \neq \langle 0,0,1\rangle$. Let $p = x_{(A\cap B)(x)}$ be the n-Cylindrical neutrosophic point of Definition 3.8(i). Then $p \neq 0_{nCy\mathcal{N}}$ and $p \in A \cap B \subseteq B$. By (ii), there exists an nCy$\mathcal{N}\mathfrak{N}$os L such that $p \in L \subseteq nCy\mathcal{N}\mathfrak{N}cl(L) \subseteq B$. Since $p \subseteq A$ and $p \subseteq L$, we have $p \subseteq A \cap L$, so $A \cap L \neq 0_{nCy\mathcal{N}}$. Also $nCy\mathcal{N}\mathfrak{N}cl(L) \subseteq B$.

(b) Let $A \neq 0_{nCy\mathcal{N}}$ be an nCy$\mathcal{N}$s and let B be an nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs with $A \cap B = 0_{nCy\mathcal{N}}$. Then $A \subseteq B^c$. Since $A \neq 0_{nCy\mathcal{N}}$, there exists $x \in U$ such that $A(x) \neq \langle 0,0,1\rangle$. Let $p = x_{A(x)}$. Then $p \neq 0_{nCy\mathcal{N}}$ and $p \in A \subseteq B^c$. Since U is StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg, there exist separated nCy$\mathcal{N}\mathfrak{N}$os's M and L such that $B \subseteq M$ and $p \in L$. Since $p \subseteq A \cap L$, we get $A \cap L \neq 0_{nCy\mathcal{N}}$. Thus L and M are separated nCy$\mathcal{N}\mathfrak{N}$os's with $A \cap L \neq 0_{nCy\mathcal{N}}$ and $B \subseteq M$.

The proofs for StnCy$\mathcal{N}\mathfrak{N}$Reg, StnCy$\mathcal{N}\mathfrak{N}\delta$Reg, StnCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$Reg, StnCy$\mathcal{N}\mathfrak{N}\delta\mathcal{S}$Reg and StnCy$\mathcal{N}\mathfrak{N}\delta\alpha$Reg spaces are similar.

Theorem 4.2 An nCy$\mathcal{N}\mathfrak{N}$ts $(U, \tau_R^{nCy\mathcal{N}}(F))$ is StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg if and only if for every nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs A in U and every n-Cylindrical neutrosophic point $x_p \in A^c$, there exist nCy$\mathcal{N}\mathfrak{N}$os's L and M in U such that $x_p \in L$, $A \subseteq M$ and nCy$\mathcal{N}\mathfrak{N}$cl(L) and nCy$\mathcal{N}\mathfrak{N}$cl(M) are separated, that is, $nCy\mathcal{N}\mathfrak{N}cl(L) \subseteq (nCy\mathcal{N}\mathfrak{N}cl(M))^c$.

Proof. Necessity. Let U be StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg, let A be an nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs in U and let $x_p \in A^c$. Then there exist separated nCy$\mathcal{N}\mathfrak{N}$os's M and L_1 such that $A \subseteq M$ and $x_p \in L_1$. Since $M \subseteq L_1^c$ and L_1^c is an nCy$\mathcal{N}\mathfrak{N}$cs, we get $nCy\mathcal{N}\mathfrak{N}cl(M) \subseteq L_1^c$, and so $x_p \in L_1 \subseteq (nCy\mathcal{N}\mathfrak{N}cl(M))^c$. Now nCy$\mathcal{N}\mathfrak{N}$cl(M) is an nCy$\mathcal{N}\mathfrak{N}$cs, and hence an nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs (see the proof of Proposition 4.1), and $x_p \in (nCy\mathcal{N}\mathfrak{N}cl(M))^c$. Since U is StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg, there exist

separated nCy$\mathcal{N}\mathfrak{N}$os's H and G such that nCy$\mathcal{N}\mathfrak{N}$cl(M) ⊆ H and $x_p \in G$. Since $G \subseteq H^c$ and H^c is an nCy$\mathcal{N}\mathfrak{N}$cs, we get nCy$\mathcal{N}\mathfrak{N}$cl(G) ⊆ H^c. Also, nCy$\mathcal{N}\mathfrak{N}$cl(M) ⊆ H implies $H^c \subseteq (\text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(M))^c$. Take L = G. Then L and M are nCy$\mathcal{N}\mathfrak{N}$os's such that $x_p \in L$, A ⊆ M and

$$\text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(L) \subseteq H^c \subseteq (\text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(M))^c,$$

that is, nCy$\mathcal{N}\mathfrak{N}$cl(L) and nCy$\mathcal{N}\mathfrak{N}$cl(M) are separated.

Sufficiency. Suppose that the condition holds. Let A be an nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs in U and let $x_p \in A^c$. Then there exist nCy$\mathcal{N}\mathfrak{N}$os's L and M such that $x_p \in L$, A ⊆ M and nCy$\mathcal{N}\mathfrak{N}$cl(L) ⊆ $(\text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(M))^c$. Since M ⊆ nCy$\mathcal{N}\mathfrak{N}$cl(M), we have $(\text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(M))^c \subseteq M^c$, and hence

$$L \subseteq \text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(L) \subseteq (\text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(M))^c \subseteq M^c.$$

Thus L and M are separated nCy$\mathcal{N}\mathfrak{N}$os's with A ⊆ M and $x_p \in L$. Hence U is StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg.

The proof for the StnCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$Reg case is similar.

Theorem 4.3 An nCy$\mathcal{N}\mathfrak{N}$ts $(U, \tau_R^{\text{nCy}\mathcal{N}}(F))$ is StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg if and only if for every nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs B in U and every nCy$\mathcal{N}$s A in U with $A \subseteq B^c$ (in particular, with $A \cap B = 0_{\text{nCy}\mathcal{N}}$), there exist separated nCy$\mathcal{N}\mathfrak{N}$os's L and M in U such that A ⊆ L and B ⊆ M.

Proof. Necessity. Let U be StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg, let B be an nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs in U and let A be an nCy$\mathcal{N}$s in U with $A \subseteq B^c$. For each $x \in U$, the n-Cylindrical neutrosophic point $x_{A(x)}$ of Definition 3.8(i) satisfies $x_{A(x)} \in A \subseteq B^c$. Since U is StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg, there exist separated nCy$\mathcal{N}\mathfrak{N}$os's L_x and M_x such that $x_{A(x)} \in L_x$, $B \subseteq M_x$ and $L_x \subseteq M_x^c$. By Definition 2.15, $\tau_R^{\text{nCy}\mathcal{N}}(F)$ has at most five members, so the family $\{(L_x, M_x) : x \in U\}$ of pairs is finite. Let $(L_1, M_1), (L_2, M_2), \dots, (L_k, M_k)$ be its distinct members, and put

$$L = \bigcup_{i=1}^{k} L_i = \bigcup_{x \in U} L_x \quad \text{and} \quad M = \bigcap_{i=1}^{k} M_i.$$

Then L and M are nCy$\mathcal{N}\mathfrak{N}$os's, since a finite intersection of nCy$\mathcal{N}\mathfrak{N}$os's is an nCy$\mathcal{N}\mathfrak{N}$os. Since $A = \bigcup_{x \in U} x_{A(x)} \subseteq \bigcup_{x \in U} L_x = L$, we have A ⊆ L, and since $B \subseteq M_i$ for every i, we have B ⊆ M. Finally, $M \subseteq M_i$ gives $M_i^c \subseteq M^c$, so $L_i \subseteq M_i^c \subseteq M^c$ for every i, and hence $L \subseteq M^c$. Thus L and M are separated nCy$\mathcal{N}\mathfrak{N}$os's with A ⊆ L and B ⊆ M. If $A \cap B = 0_{\text{nCy}\mathcal{N}}$, then $A \subseteq B^c$ (see the proof of Theorem 3.1), so the same conclusion holds.

Sufficiency. Suppose that the condition holds. Let B be an nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs in U and let $x_p \in B^c$. Then $A = x_p$ is an nCy$\mathcal{N}$s with $A \subseteq B^c$. By hypothesis, there exist separated nCy$\mathcal{N}\mathfrak{N}$os's L and M such that $x_p \in L$ and B ⊆ M. Hence U is StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg.

The proofs for StnCy$\mathcal{N}\mathfrak{N}$Reg, StnCy$\mathcal{N}\mathfrak{N}\delta$Reg, StnCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$Reg, StnCy$\mathcal{N}\mathfrak{N}\delta\mathcal{S}$Reg and StnCy$\mathcal{N}\mathfrak{N}\delta\alpha$Reg spaces are similar.

Corollary 4.1 Let $(U, \tau_R^{\text{nCy}\mathcal{N}}(F))$ be an StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg space, let B be an nCy$\mathcal{N}\mathfrak{N}\delta\beta$os in U and let A be an nCy$\mathcal{N}$s in U with A ⊆ B. Then there exists an nCy$\mathcal{N}\mathfrak{N}$ro set M such that $A \subseteq M \subseteq \text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(M) \subseteq B$.

Proof. Since B is an nCy$\mathcal{N}\mathfrak{N}\delta\beta$os, B^c is an nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs, and $A \subseteq B = (B^c)^c$. By Theorem 4.3, there exist separated nCy$\mathcal{N}\mathfrak{N}$os's L_1 and L_2 such that $A \subseteq L_1$, $B^c \subseteq L_2$ and $L_1 \subseteq L_2^c$. Since L_2^c is an nCy$\mathcal{N}\mathfrak{N}$cs containing L_1, we get $\text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(L_1) \subseteq L_2^c$, and since $B^c \subseteq L_2$, we have $L_2^c \subseteq B$. Hence

$$\text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(L_1) \subseteq L_2^c \subseteq B. \quad (1)$$

Put $M = \text{nCy}\mathcal{N}\mathfrak{N}\text{int}(\text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(L_1))$. Since $M \subseteq \text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(L_1)$, we have $\text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(M) \subseteq \text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(\text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(L_1)) = \text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(L_1)$. Since L_1 is an nCy$\mathcal{N}\mathfrak{N}$os and $L_1 \subseteq \text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(L_1)$, we have

$$L_1 = \text{nCy}\mathcal{N}\mathfrak{N}\text{int}(L_1) \subseteq \text{nCy}\mathcal{N}\mathfrak{N}\text{int}(\text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(L_1)) = M, \quad (2)$$

and so $\text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(L_1) \subseteq \text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(M)$. Hence $\text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(M) = \text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(L_1)$, and therefore

$$\text{nCy}\mathcal{N}\mathfrak{N}\text{int}(\text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(M)) = \text{nCy}\mathcal{N}\mathfrak{N}\text{int}(\text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(L_1)) = M,$$

that is, M is an nCy$\mathcal{N}\mathfrak{N}$ro set. Finally, by (2) and (1),

$$A \subseteq L_1 \subseteq M \subseteq \text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(M) = \text{nCy}\mathcal{N}\mathfrak{N}\text{cl}(L_1) \subseteq B.$$

The proofs for StnCy$\mathcal{N}\mathfrak{N}$Reg, StnCy$\mathcal{N}\mathfrak{N}\delta$Reg, StnCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$Reg, StnCy$\mathcal{N}\mathfrak{N}\delta\mathcal{S}$Reg and StnCy$\mathcal{N}\mathfrak{N}\delta\alpha$Reg spaces are similar.

Theorem 4.4 Let $(U_1, \tau_R^{\text{nCy}\mathcal{N}}(F_1))$ and $(U_2, \tau_R^{\text{nCy}\mathcal{N}}(F_2))$ be two nCy$\mathcal{N}\mathfrak{N}$ts's and let $h_{\mathcal{C}} : U_1 \to U_2$ be a bijection. If $h_{\mathcal{C}}$ is nCy$\mathcal{N}\mathfrak{N}\delta\beta$Irr (resp. nCy$\mathcal{N}\mathfrak{N}Cts, nCy\mathcal{N}\mathfrak{N}\delta$Irr, nCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}Irr, nCy\mathcal{N}\mathfrak{N}\delta\mathcal{S}$Irr and nCy$\mathcal{N}\mathfrak{N}\delta\alpha$Irr) and nCy$\mathcal{N}\mathfrak{N}$O, and U_1 is StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg (resp. StnCy$\mathcal{N}\mathfrak{N}$Reg, StnCy$\mathcal{N}\mathfrak{N}\delta$Reg, StnCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$Reg, StnCy$\mathcal{N}\mathfrak{N}\delta\mathcal{S}$Reg and StnCy$\mathcal{N}\mathfrak{N}\delta\alpha$Reg), then U_2 is StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg (resp. StnCy$\mathcal{N}\mathfrak{N}$Reg, StnCy$\mathcal{N}\mathfrak{N}\delta$Reg, StnCy$\mathcal{N}\mathfrak{N}\delta\mathcal{P}$Reg, StnCy$\mathcal{N}\mathfrak{N}\delta\mathcal{S}$Reg and StnCy$\mathcal{N}\mathfrak{N}\delta\alpha$Reg).

Proof. As in the proof of Theorem 3.4, since $h_{\mathcal{C}}$ is a bijection, for nCy$\mathcal{N}$s's S, T in U_1 and A in U_2 we have: $S \subseteq T$ implies $h_{\mathcal{C}}(S) \subseteq h_{\mathcal{C}}(T)$, $h_{\mathcal{C}}(S^c) = (h_{\mathcal{C}}(S))^c$, $h_{\mathcal{C}}(h_{\mathcal{C}}^{-1}(A)) = A$ and $h_{\mathcal{C}}(x_p) = (h_{\mathcal{C}}(x))_p$ for every n-Cylindrical neutrosophic point x_p in U_1.

Suppose that U_1 is StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg. Let A be an nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs in U_2 and let $y_p \in A^c$, where $p = \langle a, b, c \rangle$. Since $h_{\mathcal{C}}$ is nCy$\mathcal{N}\mathfrak{N}\delta\beta$Irr, $h_{\mathcal{C}}^{-1}(A)$ is an nCy$\mathcal{N}\mathfrak{N}\delta\beta$cs in U_1 (Definition 3.9). Since $h_{\mathcal{C}}$ is a surjection, there exists $x \in U_1$ such that $h_{\mathcal{C}}(x) = y$. Since $y_p \in A^c$, we have $a \le \gamma_A(h_{\mathcal{C}}(x))$, $b \le 1 - \beta_A(h_{\mathcal{C}}(x))$ and $c \ge \alpha_A(h_{\mathcal{C}}(x))$, that is, $x_p \in (h_{\mathcal{C}}^{-1}(A))^c$. Since U_1 is StnCy$\mathcal{N}\mathfrak{N}\delta\beta$Reg, there exist separated nCy$\mathcal{N}\mathfrak{N}$os's L and M in U_1 such that $x_p \in L$, $h_{\mathcal{C}}^{-1}(A) \subseteq M$ and $L \subseteq M^c$. Since $h_{\mathcal{C}}$ is nCy$\mathcal{N}\mathfrak{N}$O, $h_{\mathcal{C}}(L)$ and $h_{\mathcal{C}}(M)$ are nCy$\mathcal{N}\mathfrak{N}$os's in U_2. Now

$$y_p = h_{\mathcal{C}}(x_p) \subseteq h_{\mathcal{C}}(L) \quad \text{and} \quad A = h_{\mathcal{C}}(h_{\mathcal{C}}^{-1}(A)) \subseteq h_{\mathcal{C}}(M),$$

that is, $y_p \in h_{\mathcal{C}}(L)$ and $A \subseteq h_{\mathcal{C}}(M)$. Also, $L \subseteq M^c$ implies $h_{\mathcal{C}}(L) \subseteq h_{\mathcal{C}}(M^c) = (h_{\mathcal{C}}(M))^c$, so $h_{\mathcal{C}}(M)$ and $h_{\mathcal{C}}(L)$ are separated. Hence U_2 is $\mathrm{StnCy}\mathcal{N}\mathfrak{N}\delta\beta\mathrm{Reg}$.

The proofs of the other cases are similar.

Remark 4.2 Theorems 4.1, 4.3, 4.4 and Corollary 4.1 have been proved for $\mathrm{StnCy}\mathcal{N}\mathfrak{N}\delta\beta\mathrm{Reg}$ spaces, and they also hold for $\mathrm{StnCy}\mathcal{N}\mathfrak{N}\mathrm{Reg}$, $\mathrm{StnCy}\mathcal{N}\mathfrak{N}\delta\mathrm{Reg}$, $\mathrm{StnCy}\mathcal{N}\mathfrak{N}\delta\mathcal{P}\mathrm{Reg}$, $\mathrm{StnCy}\mathcal{N}\mathfrak{N}\delta\mathcal{S}\mathrm{Reg}$ and $\mathrm{StnCy}\mathcal{N}\mathfrak{N}\delta\alpha\mathrm{Reg}$ spaces, since their proofs use only the definition of the space together with properties (i) and (ii) of Remark 3.1 On the other hand, Propositions 4.1, 4.2 and Theorem 4.2 have been stated only for the $\delta\beta$ and δ pre cases. Their proofs use the fact that every $\mathrm{nCy}\mathcal{N}\mathfrak{N}\mathrm{os}$ is an $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\beta\mathrm{os}$ and an $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\mathcal{P}\mathrm{os}$ (equivalently, every $\mathrm{nCy}\mathcal{N}\mathfrak{N}\mathrm{cs}$ is an $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\beta\mathrm{cs}$ and an $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\mathcal{P}\mathrm{cs}$), and an $\mathrm{nCy}\mathcal{N}\mathfrak{N}\mathrm{os}$ need not be an $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\mathrm{os}$, an $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\mathcal{S}\mathrm{os}$ or an $\mathrm{nCy}\mathcal{N}\mathfrak{N}\delta\alpha\mathrm{os}$ (Example 4.2(i)).

5 An application to the selection of nanocarriers in drug delivery

In nanotechnology based drug delivery, a drug is loaded into a nanocarrier, such as a liposome, a polymeric nanoparticle, a solid lipid nanoparticle or a dendrimer, so that it reaches the target tissue in the required amount and is released over the required period. The performance of a formulation depends on several physicochemical attributes, for example the particle size, the surface charge, the release profile and the biocompatibility of the carrier material, and the preclinical evaluation of a formulation is usually uncertain. For each formulation x, a panel of experts assesses the property "x carries a therapeutic risk" (toxicity, adverse effects, premature release or rapid clearance) by three degrees: the degree $\alpha(x)$ of evidence for the risk, the degree $\gamma(x)$ of evidence against it (that is, evidence that x is safe and effective), and the degree $\beta(x)$ of indeterminacy (inconclusive or inconsistent test results). The degrees α and γ are dependent, since strong evidence for a risk limits the evidence against it, whereas β is independent of them. Moreover, experts often give pairs (α, γ) with $\alpha + \gamma > 1$ or even $\alpha^2 + \gamma^2 >$ (i) Hence such evaluations are naturally modelled by n-Cylindrical neutrosophic sets. Following the approach of Lellis Thivagar et al. [7, 8], which goes back to the rough sets of Pawlak [13], we use the n-Cylindrical neutrosophic nano topology of Definition 2.15 to find the attributes that are decisive for the risk of a formulation, and then select the most suitable formulation.

Algorithm. [Step (i)]

(i) Let U be the set of candidate formulations, C the set of condition attributes, and F the $\mathrm{nCy}\mathcal{N}$s on U that gives the expert evaluation $\langle \alpha_F(x), \beta_F(x), \gamma_F(x) \rangle$ of the risk of each formulation x. The integer n is fixed by the evaluation scale used by the experts, before the data are collected; if no scale is prescribed, the smallest integer $n \geq 2$ such that $\alpha_F^n(x) + \gamma_F^n(x) \leq 1$ for all $x \in U$ may be used as a default. Once chosen, n is kept fixed throughout the procedure.

(ii) For $P \subseteq C$, let R_P be the equivalence relation on U defined by xR_Py if and only if $a(x) = a(y)$ for every $a \in P$.

(iii) Compute $\underline{\mathrm{nCy}\mathcal{N}}(F)$, $\overline{\mathrm{nCy}\mathcal{N}}(F)$ and $B_{\mathrm{nCy}\mathcal{N}}(F)$ with respect to R_C by Definition 2.14, and check that

$$\tau_C(F) = \{0_{\mathrm{nCy}\mathcal{N}}, 1_{\mathrm{nCy}\mathcal{N}}, \underline{\mathrm{nCy}\mathcal{N}}(F), \overline{\mathrm{nCy}\mathcal{N}}(F), B_{\mathrm{nCy}\mathcal{N}}(F)\}$$

is an $\mathrm{nCy}\mathcal{N}\mathrm{t}$, so that $(U, \tau_C(F))$ is an $\mathrm{nCy}\mathcal{N}\mathfrak{N}\mathrm{ts}$.

(iv) For each $a \in C$, compute $\tau_{C-\{a\}}(F)$ in the same way, using $R_{C-\{a\}}$. If $\tau_{C-\{a\}}(F) = \tau_C(F)$, then a is not needed to describe F; otherwise a is called a core attribute. Let

$$\mathrm{CORE}(C) = \{a \in C : \tau_{C-\{a\}}(F) \neq \tau_C(F)\}.$$

(v) Compute $\overline{\mathrm{nCy}\mathcal{N}}(F)$ with respect to $R_{\mathrm{CORE}(C)}$. For $x \in U$, the triple $\overline{\mathrm{nCy}\mathcal{N}}(F)(x)$ is the worst-case evaluation (largest evidence for the risk, smallest evidence against it) among the formulations that agree with x on the core attributes. Rank the classes of $R_{\mathrm{CORE}(C)}$ by the score

$$s_n(\langle \alpha, \beta, \gamma \rangle) = \gamma^n - \alpha^n$$

of their worst-case evaluations, and, within the best class, rank the formulations by the score $s_n(F(x))$ of their own evaluations. A larger score means a safer formulation; for $n = 3$, $s_3(\langle \alpha, \beta, \gamma \rangle)$ is the score function of Senapati and Yager [17] of the Fermatean pair (γ, α), which belongs to the complement $\langle \gamma, 1 - \beta, \alpha \rangle$, that is, to the property "x is safe".

Example 5.1 A pharmaceutical research team compares six nanocarrier formulations $U = \{D_1, D_2, D_3, D_4, D_5, D_6\}$ of an anticancer drug. The condition attributes are $C = \{a_1, a_2, a_3, a_4\}$, where a_1 is the particle size (small, below 100 nm; medium, 100–200 nm; large, above 200 nm), a_2 is the surface charge (neutral, negative or positive), a_3 is the drug release profile (controlled or burst) and a_4 is the biocompatibility of the carrier material (high or low). The attribute values and the evaluations of the risk of each formulation by a panel of experts, based on preclinical tests, are given in Table 5.1

Formulation	a_1	a_2	a_3	a_4	$F = \langle \alpha_F, \beta_F, \gamma_F$
D_1	small	neutral	controlled	high	⟨0.40,0.20,0.
D_2	small	neutral	burst	high	⟨0.60,0.40,0.
D_3	medium	negative	controlled	high	⟨0.50,0.30,0.
D_4	medium	negative	controlled	low	⟨0.70,0.45,0.
D_5	large	positive	burst	low	⟨0.75,0.50,0.
D_6	small	neutral	controlle	high	⟨0.45,0.25,0.

			d		

Table 1: Attributes and expert evaluations of the risk of the formulations.

Step (i) Since $0.40^2 + 0.95^2 = (\text{i})0625$, $0.70^2 + 0.75^2 = (\text{i})0525$, $0.75^2 + 0.80^2 = (\text{i})2025$ and $0.45^2 + 0.90^2 = (\text{i})0125$, the evaluations of D_1, D_4, D_5 and D_6 are not Pythagorean, while

$$0.40^3 + 0.95^3 = 0.9214, 0.60^3 + 0.75^3 = 0.6379, 0.50^3 + 0.85^3 = 0.7391,$$
$$0.70^3 + 0.75^3 = 0.7649, 0.75^3 + 0.80^3 = 0.9339, 0.45^3 + 0.90^3 = 0.8201$$

are all at most (i) Hence the evaluations are given on a Fermatean ($n = 3$) scale, which is also the smallest admissible value, and F is a 3-Cylindrical neutrosophic set on U.

Step (ii) The equivalence classes of R_C are $\{D_1, D_6\}$, $\{D_2\}$, $\{D_3\}$, $\{D_4\}$ and $\{D_5\}$.

Step (iii) By Definition 2.14, for the class $\{D_1, D_6\}$ we obtain the lower approximation $\langle \min\{0.40,0.45\}, \min\{0.20,0.25\}, \max\{0.95,0.90\}\rangle = \langle 0.40,0.20,0.95\rangle$ and the upper approximation $\langle 0.45,0.25,0.90\rangle$, and the boundary is

$$\langle 0.45,0.25,0.90\rangle \cap \langle 0.40,0.20,0.95\rangle^c = \langle 0.45,0.25,0.90\rangle \cap \langle 0.95,0.80,0.40\rangle = \langle 0.45,0.25,0.90\rangle.$$

For the other classes, which are singletons, the lower and upper approximations coincide with F, and the same computation shows that the boundary also coincides with F. Thus

$$\underline{\mathrm{nCy}\mathcal{N}}(F) = \{\langle D_1, 0.40,0.20,0.95\rangle, \langle D_2, 0.60,0.40,0.75\rangle, \langle D_3, 0.50,0.30,0.85\rangle,$$
$$\langle D_4, 0.70,0.45,0.75\rangle, \langle D_5, 0.75,0.50,0.80\rangle, \langle D_6, 0.40,0.20,0.95\rangle\},$$

$$\overline{\mathrm{nCy}\mathcal{N}}(F) = B_{\mathrm{nCy}\mathcal{N}}(F) = \{\langle D_1, 0.45,0.25,0.90\rangle, \langle D_2, 0.60,0.40,0.75\rangle, \langle D_3, 0.50,0.30,0.85\rangle,$$
$$\langle D_4, 0.70,0.45,0.75\rangle, \langle D_5, 0.75,0.50,0.80\rangle, \langle D_6, 0.45,0.25,0.90\rangle\}.$$

Since $0_{\mathrm{nCy}\mathcal{N}} \subseteq \underline{\mathrm{nCy}\mathcal{N}}(F) \subseteq \overline{\mathrm{nCy}\mathcal{N}}(F) \subseteq 1_{\mathrm{nCy}\mathcal{N}}$, the family

$$\tau_C(F) = \{0_{\mathrm{nCy}\mathcal{N}}, 1_{\mathrm{nCy}\mathcal{N}}, \underline{\mathrm{nCy}\mathcal{N}}(F), \overline{\mathrm{nCy}\mathcal{N}}(F)\}$$

is a chain, hence closed under unions and intersections, and $(U, \tau_C(F))$ is an nCy$\mathcal{N}\Re$ts.

Step (iv) Removing one attribute at a time, we obtain the partitions in Table (v)1; in each case the lower approximation, the upper approximation and the boundary again form a chain, so each $\tau_{C-\{a\}}(F)$ is an nCy$\mathcal{N}\Re$ topology.

P	Classes of R_P	$\tau_P(F) = \tau_C(F)$?
$C - \{a_1\}$	$\{D_1, D_6\}, \{D_2\}, \{D_3\}, \{D_4\}, \{D_5\}$	yes
$C - \{a_2\}$	$\{D_1, D_6\}, \{D_2\}, \{D_3\}, \{D_4\}, \{D_5\}$	yes
$C - \{a_3\}$	$\{D_1, D_2, D_6\}, \{D_3\}, \{D_4\}, \{D_5\}$	no
$C - \{a_4\}$	$\{D_1, D_6\}, \{D_2\}, \{D_3, D_4\}, \{D_5\}$	no

Table 2: Partitions obtained by removing one attribute.

For $P = C - \{a_1\}$ and $P = C - \{a_2\}$, the partition, and hence the topology, is unchanged. For $P = C - \{a_3\}$, the formulations D_1, D_2 and D_6 become indistinguishable, and the upper approximation at D_1 becomes

$$\langle \max\{0.40,0.60,0.45\}, \max\{0.20,0.40,0.25\}, \min\{0.95,0.75,0.90\}\rangle = \langle 0.60,0.40,0.75\rangle \neq \langle 0.45,0.25,0.90\rangle,$$

so $\tau_{C-\{a_3\}}(F) \neq \tau_C(F)$. For $P = C - \{a_4\}$, the formulations D_3 and D_4 become indistinguishable, and the upper approximation at D_3 becomes $\langle 0.70,0.45,0.75\rangle \neq \langle 0.50,0.30,0.85\rangle$, so $\tau_{C-\{a_4\}}(F) \neq \tau_C(F)$. Therefore

$$\mathrm{CORE}(C) = \{a_3, a_4\},$$

that is, the release profile and the biocompatibility of the carrier are the decisive attributes, while the particle size and the surface charge are not needed to describe the risk of these formulations.

Step (v) The classes of $R_{\mathrm{CORE}(C)}$ are $\{D_1, D_3, D_6\}$ (controlled release, high biocompatibility), $\{D_2\}$ (burst release, high biocompatibility), $\{D_4\}$ (controlled release, low biocompatibility) and $\{D_5\}$ (burst release, low biocompatibility). Their worst-case evaluations and scores are given in Table 5.1

Class	$\overline{\mathrm{nCy}\mathcal{N}}(F)$ w.r.t. $R_{\mathrm{CORE}(C)}$	s_3
$\{D_1, D_3, D_6\}$	$\langle 0.50,0.30,0.85\rangle$	0.4891
$\{D_2\}$	$\langle 0.60,0.40,0.75\rangle$	0.2059
$\{D_4\}$	$\langle 0.70,0.45,0.75\rangle$	0.0789
$\{D_5\}$	$\langle 0.75,0.50,0.80\rangle$	0.0901

Table 3: Worst-case evaluations of the classes of $R_{\mathrm{CORE}(C)}$.

The class of formulations with controlled release and high biocompatibility has the highest worst-case score. Within this class, the scores of the individual evaluations are

$$s_3(F(D_1)) = 0.95^3 - 0.40^3 = 0.7934,$$
$$s_3(F(D_6)) = 0.90^3 - 0.45^3 = 0.6379,$$
$$s_3(F(D_3)) = 0.85^3 - 0.50^3 = 0.4891,$$

so the ranking is $D_1 \succ D_6 \succ D_3 \succ D_2 \succ D_5 \succ D_4$, and the formulation D_1 (small, neutral, controlled release, highly biocompatible carrier) is recommended for further development.

Remark 5.1 (i) The core attributes are found without any weights: an attribute is decisive exactly when ignoring it changes the n-Cylindrical neutrosophic nano topology generated by F. In practice, this tells the formulation scientist which properties must be controlled

most carefully during development.

(ii) The evaluations of D_1, D_4, D_5 and D_6 are not 2-Cylindrical neutrosophic numbers, since $\alpha^2+\gamma^2>1$; with $n=3$, they are admissible without changing the experts' values, and the independent component β records the inconclusive test results separately.

(iii) In general, the family $\{0_{nCy\mathcal{N}}, 1_{nCy\mathcal{N}}, \underline{nCy\mathcal{N}}(F), \overline{nCy\mathcal{N}}(F), B_{nCy\mathcal{N}}(F)\}$ need not be closed under unions and intersections, because, by Remark (ii)2, $\underline{nCy\mathcal{N}}(F) \cap (\underline{nCy\mathcal{N}}(F))^c$ need not be $0_{nCy\mathcal{N}}$; therefore Step 3 must be checked. It holds, for instance, whenever $\alpha_{\underline{R}(F)}(x) \le \gamma_{\underline{R}(F)}(x)$ and $\beta_{\underline{R}(F)}(x) \le \frac{1}{2}$ for all $x \in U$, since then $\underline{nCy\mathcal{N}}(F) \subseteq B_{nCy\mathcal{N}}(F) \subseteq \overline{nCy\mathcal{N}}(F)$; this is the case in Example (v)1 and for every coarser partition.

(iv) Sensitivity. We changed each of the 18 degrees in Table (v)1 by ± 0.05, one at a time. In 32 of the 36 cases the evaluations remain admissible and satisfy the condition in (iii). In all of these cases $CORE(C)=\{a_3,a_4\}$, the formulation D_1 is still selected, and the first four places $D_1 \succ D_6 \succ D_3 \succ D_2$ of the ranking are unchanged; only D_4 and D_5 may exchange the last two places. Thus the decision is stable under small changes of the experts' evaluations.

Remark 5.2 We now relate Example (v)1 to the regular spaces of Sections 3 and 4 Write $L=\underline{nCy\mathcal{N}}(F)$ and $H=\overline{nCy\mathcal{N}}(F)$, computed with respect to R_C, so that $\tau_C(F)=\{0_{nCy\mathcal{N}}, 1_{nCy\mathcal{N}}, L, H\}$ with $L \subseteq H$ and $L \neq H$. At every formulation, the values of H satisfy $\alpha_H \le \gamma_H$ and $\beta_H \le \frac{1}{2}$, so $H \subseteq H^c$, and hence $0_{nCy\mathcal{N}} \subseteq L \subseteq H \subseteq H^c \subseteq L^c \subseteq 1_{nCy\mathcal{N}}$.

The $nCy\mathcal{N}\mathfrak{N}cs$'s are $0_{nCy\mathcal{N}}$, H^c, L^c and $1_{nCy\mathcal{N}}$. Therefore $nCy\mathcal{N}\mathfrak{N}cl(L)=nCy\mathcal{N}\mathfrak{N}cl(H)=H^c$ and $nCy\mathcal{N}\mathfrak{N}int(H^c)=nCy\mathcal{N}\mathfrak{N}int(L^c)=H$. It follows that the $nCy\mathcal{N}\mathfrak{N}ro$ sets are $0_{nCy\mathcal{N}}$, H and $1_{nCy\mathcal{N}}$, and the $nCy\mathcal{N}\mathfrak{N}rc$ sets are $0_{nCy\mathcal{N}}$, H^c and $1_{nCy\mathcal{N}}$; in particular, the lower approximation L is an $nCy\mathcal{N}\mathfrak{N}os$ but not an $nCy\mathcal{N}\mathfrak{N}\delta os$.

(i) Every $nCy\mathcal{N}s$ S in U is an $nCy\mathcal{N}\mathfrak{N}\delta\beta os$: if $0_{nCy\mathcal{N}} \neq S \subseteq H^c$, then $nCy\mathcal{N}\mathfrak{N}\delta cl(S)=H^c$ and $S \subseteq H^c = nCy\mathcal{N}\mathfrak{N}cl(H)=nCy\mathcal{N}\mathfrak{N}cl(nCy\mathcal{N}\mathfrak{N}int(H^c))$; otherwise $nCy\mathcal{N}\mathfrak{N}\delta cl(S) \in \{0_{nCy\mathcal{N}}, 1_{nCy\mathcal{N}}\}$ and the condition is clear. Consequently, as in Example 4.1, $(U,\tau_C(F))$ is $nCy\mathcal{N}\mathfrak{N}\delta\beta Reg$ and $St^*nCy\mathcal{N}\mathfrak{N}\delta\beta Reg$, while, by Proposition 4.3, it is not $StnCy\mathcal{N}\mathfrak{N}\delta\beta Reg$.

(ii) $(U,\tau_C(F))$ is not $nCy\mathcal{N}\mathfrak{N}Reg$: let $A=H^c$ and let p be the n-Cylindrical neutrosophic point at D_2 with value $\langle 0.50,0.30,0.80\rangle$. Since $H(D_2)=\langle 0.60,0.40,0.75\rangle$, we have $p \in H = A^c$. Since $H^c(D_2)=\langle 0.75,0.60,0.60\rangle \not\supseteq H(D_2)$, the only $nCy\mathcal{N}\mathfrak{N}os$ containing A is $1_{nCy\mathcal{N}}$, and then no $nCy\mathcal{N}\mathfrak{N}os$ separated from it can contain p.

Thus the nano topology of the application is $\delta\beta$-regular but not regular. In terms of the application, a group of formulations described by an $nCy\mathcal{N}\mathfrak{N}\delta\beta cs$ and the evaluation of a single formulation outside it can always be separated by $nCy\mathcal{N}\mathfrak{N}\delta\beta os$'s, although the four $nCy\mathcal{N}\mathfrak{N}os$'s, which are built only from the approximations of F, are too coarse to separate them. This illustrates why the $\delta\beta$-versions of regularity are the appropriate ones for nano topologies arising from data.

6. Conclusion

In this paper, we have introduced and studied n-Cylindrical neutrosophic nano (resp. δ, δ pre, δ semi, $\delta\alpha$ and $\delta\beta$) regular spaces, together with their strongly and strongly * versions, in the framework of $nCy\mathcal{N}\mathfrak{N}ts$'s. Since the intersection of an $nCy\mathcal{N}s$ and its complement need not be $0_{nCy\mathcal{N}}$, these spaces have been defined by means of n-Cylindrical neutrosophic points and separated $nCy\mathcal{N}s$'s rather than disjoint ones. We have characterized $nCy\mathcal{N}\mathfrak{N}\delta\beta Reg$ spaces by means of $nCy\mathcal{N}\mathfrak{N}\delta\beta$ closures and $nCy\mathcal{N}\mathfrak{N}\delta\beta nbhd$'s, shown that every $nCy\mathcal{N}\mathfrak{N}\delta\beta os$ in such a space is a union of $nCy\mathcal{N}\mathfrak{N}\delta\beta cs$'s, and shown that this regularity is preserved by $nCy\mathcal{N}\mathfrak{N}\delta\beta Irr$ and $nCy\mathcal{N}\mathfrak{N}\delta\beta O$ bijections and by suitable continuous, closed or open maps. For strongly regular spaces, we have obtained analogous characterizations by means of $nCy\mathcal{N}\mathfrak{N}$ closures and closed neighbourhoods, shown that an $nCy\mathcal{N}\mathfrak{N}ro$ set can be inserted between an $nCy\mathcal{N}s$ and an $nCy\mathcal{N}\mathfrak{N}\delta\beta os$ containing it, and established the relations among strongly, strongly * and ordinary regular spaces; Examples (iv)1 and (iv)2 show that the converses of Proposition 4.1 and of Proposition 4.2(ii) need not hold, Example 4.3 gives a $\delta\beta$-regular space in which not every set is $\delta\beta$-open, and Proposition 4.3 shows that strong $\delta\beta$-regularity is a very restrictive condition.

As a real life application, we have used n-Cylindrical neutrosophic nano topology in drug delivery technology. The risk of each nanocarrier formulation was evaluated as an n-Cylindrical neutrosophic number, which allows the experts' evidence for and against the risk to exceed the intuitionistic and Pythagorean bounds and records inconclusive test results separately. By comparing the n-Cylindrical neutrosophic nano topologies generated by the lower and upper approximations and the boundary, the proposed algorithm identified the release profile and the biocompatibility of the carrier as the core attributes, without any weights, and the score function selected the most suitable formulation; this decision is stable under small changes of the data. We have also given a condition under which the lower and upper approximations and the boundary form an n-Cylindrical neutrosophic nano topology, and we have shown that the nano topology of the application is $\delta\beta$-regular but not regular, which illustrates the role of the $\delta\beta$-versions of regularity for nano topologies arising from data.

The numerical example uses illustrative data. In future work, the application may be tested with real experimental data and with more attributes and experts, and the ideas of this paper may be extended to normal spaces and other separation axioms in n-Cylindrical neutrosophic nano topological spaces, and to other problems of medical decision making.

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