Application Of Decomposition Of Cartesian Product And Corona Product Of Sunlet And Cycle In Drug Deliverance.

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ABSTRACT
For any positive integer k≥3, we define the sunlet graph of order 2k, denoted by L_2k, as the graph consisting of a cycle of length k together with k pendant vertices such that each pendant vertex is adjacent to exactly one vertex of the cycle so that the degree of each vertex in the cycle is 3. In this paper, we show the necessary and sufficient condition for the decomposition of cartesian product and corona product of sunlet and cycle into paws, stars, cycles, claws and paths.
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INTRODUCTION

All graphs considered here are finite and undirected unless otherwise stated. Let C_r be the cycle on r vertices. For a graph G, if $E(G)$ can be partitioned into $E_1, E_2, \ldots, E_m$ such that the subgraph of G induced by E_i is H_i for all $1 \leq i \leq m$, then $H_1, H_2, \ldots, H_m$ decompose G and is written as $E(G) = E(H_1) \cup E(H_2) \cup \ldots \cup E(H_m)$. If for $1 \leq i \leq m$, $H_i \cong H$, we say that G has a H-decomposition. Numerous studies have addressed the decomposition of graphs in various contexts. In [4], S. Arumugam, I. Sahul Hamid and V. M. Abraham gave the decomposition of a graph G on n vertices (not necessarily connected) into $\lfloor n/2 \rfloor$ paths and cycles. For any two graphs G and H, their cartesian product, denoted by $G \square H$ has vertex set $V(G \square H) = V(G) \times V(H)$ and edge set $E(G \square H) = \{(g,h)(g',h'): g = g',\ hh' \in E(H),\ or\ gg' \in E(G),\ h = h'\}$. R. Frucht and F. Harary in their paper [3], introduced a new and simple operation on two graphs G_1 and G_2 called their corona product. For any two graphs G_1 and G_2 of orders m and n respectively, their corona product $G_1 \circledcirc G_2$ is the graph obtained by taking one copy of G_1 and m copies of G_2 such that the i^{th} vertex of G_1 is connected to every vertex in the i^{th} copy of G_2. A. Muthusamy and K. Sowndhariya [1], studied the cartesian product of complete graphs into sunlet graphs of order eight. Here in this paper, we focus on the decomposition of cartesian product and corona product of sunlet and cycle.

Remark 1.1.

Consider a graph H_1 with $V(H_1) = \{x_{i,j}: 1 \leq i \leq 6;\ 1 \leq j \leq 3\}$ and

$E(H_1) = \{x_{i,j}x_{i,j+1}, x_{k,l}x_{k+1,l}, x_{1,q}x_{3,q}, x_{r,s}x_{r+3,s}: 1 \leq i \leq 6;\ 1 \leq j \leq 2;\ 1 \leq k \leq 5;\ 1 \leq l, q, r, s \leq 3\}$. H_1 is decomposed into 4 copies of $K_{1,3}$ and 6 copies of $K_{1,2}$ as follows:

$\{x_{1,2}x_{1,1}, x_{1,2}x_{2,2}, x_{1,2}x_{3,2}\}, \{x_{1,3}x_{1,2}, x_{1,3}x_{2,3}, x_{1,3}x_{3,3}\}, \{x_{2,2}x_{2,1}, x_{2,2}x_{3,2}, x_{2,2}x_{5,2}\}$,

$\{x_{2,3}x_{2,2}, x_{2,3}x_{3,3}, x_{2,3}x_{5,3}\}, \{x_{3,2}x_{3,1}, x_{3,2}x_{6,2}\}$, $\{x_{3,3}x_{3,2}, x_{3,3}x_{6,3}\}, \{x_{4,2}x_{4,1}, x_{4,2}x_{1,2}\}$, $\{x_{4,3}x_{4,2}, x_{4,3}x_{1,3}\}, \{x_{5,2}x_{5,1}, x_{5,2}x_{5,3}\}$ and $\{x_{6,2}x_{6,1}, x_{6,2}x_{6,3}\}$.

Theorem 1.2. *For any two positive integers m, n with $m = 3$, $n \geq 3$ and n is odd,*

$L_{2m} \square C_n$ is decomposed into p copies of $K_{1,3}$ and q copies of $K_{1,2}$ if and only if

$p + \frac{2}{3}q = \frac{m(5n+1)}{3} - (n+1)$.

Proof:

Let $G = L_{2m} \square C_n$, where $V(G) = \{x_{i,j}: 1 \leq i \leq 2m;\ 1 \leq j \leq n\}$ and

$E(G) = \{x_{i,j}x_{i,j+1}, x_{k,1}x_{k,1+n}, x_{p,q}x_{p+1,q}, x_{1,s}x_{m,s}, x_{u,v}x_{u+m,v}: 1 \leq i \leq 2m;$

$1 \leq j \leq n-1;\ 1 \leq k \leq 2m;\ 1 \leq p \leq m-1;\ 1 \leq q, s, v \leq n;\ 1 \leq u \leq m\}$.

Suppose that $L_{2m} \square C_n$ is decomposed into p copies of $K_{1,3}$ and q copies of $K_{1,2}$.

Since $|E(G)| = 3p + 2q$, we have

$3p + 2q = 3[(m-1)(n+1)] + 2[m(n-1)]$.

$\Rightarrow 3p + 2q = 5mn + m - 3n - 3.$

Therefore, $p + \frac{2}{3}q = \frac{m(5n+1)}{3} - (n+1)$.

Conversely, suppose that $p + \frac{2}{3}q = \frac{m(5n+1)}{3} - (n+1)$.

For $n = 3$, the set of edges $\{x_{1,1}x_{1,2}, x_{1,1}x_{2,1}, x_{1,1}x_{1,3}\}, \{x_{2,2}x_{1,2}, x_{2,2}x_{2,1}, x_{2,2}x_{5,2}\}$,

$\{x_{2,3}x_{2,2}, x_{2,3}x_{2,1}, x_{2,3}x_{3,3}\}, \{x_{3,3}x_{1,3}, x_{3,3}x_{3,2}, x_{3,3}x_{3,1}\}, \{x_{3,1}x_{1,1}, x_{3,1}x_{2,1}, x_{3,1}x_{3,2}\}$,

$\{x_{3,2}x_{2,2}, x_{3,2}x_{1,2}, x_{3,2}x_{6,2}\}, \{x_{5,3}x_{5,2}, x_{5,3}x_{5,1}, x_{5,3}x_{1,3}\}$ and $\{x_{6,1}x_{6,2}, x_{6,1}x_{3,1}, x_{6,1}x_{6,3}\}$ forms 8 copies of $K_{1,3}$. Also, the set of edges $\{x_{4,1}x_{1,1}, x_{4,1}x_{4,3}\}, \{x_{1,3}x_{1,2}, x_{1,3}x_{2,3}\}$,

$\{x_{5,1}x_{2,1}, x_{5,1}x_{5,2}\}, \{x_{4,2}x_{4,1}, x_{4,2}x_{1,2}\}, \{x_{4,3}x_{4,2}, x_{4,3}x_{1,3}\}$ an d $\{x_{6,3}x_{6,2}, x_{6,3}x_{3,3}\}$ forms 6 copies of $K_{1,2}$. Therefore $L_6 \square P_3$ is decomposed into 8 copies of $K_{1,3}$ and 6 copies of $K_{1,2}$.

For $n \geq 5$, we prove this theorem by induction on n.

For $n = 5$, the induced subgraph $< \{x_{i,4}, x_{i,5}\} >$ together with the edge $x_{i,3}x_{i,4}$ forms H_1

for all $1 \leq i \leq 2m$. Thus $E(L_{2m} \square C_5) = E(L_{2m} \square C_3) \cup E(H_1)$.

Assume that the theorem is true for $n - 2$.

The induced subgraph $< \{x_{i,j}, x_{i,j+1}\} >$ together with the edge $x_{i,j-1}x_{i,j}$ where

$j = 6, 8, \ldots, n-1$ forms H_1 $\forall\, 1 \leq i \leq 2m$. Thus $E(L_{2m} \square C_n) = E(L_{2m} \square C_{n-2}) \cup E(H_1)$.

Therefore by induction hypothesis, $L_{2m} \square C_{n-2}$ is decomposed into $2(n-1)$ copies of $K_{1,3}$ and $3(n-3)$ copies of $K_{1,2}$. Also by Remark 1.1., H_1 is decomposed into 4 copies of $K_{1,3}$ and 6 copies of $K_{1,2}$. Hence $L_{2m} \square C_n$ is decomposed into p copies of $K_{1,3}$ and q copies of $K_{1,2}$.

Remark 1.3.

Consider a graph H_2 with $V(H_2) = \{x_{i,j}: 1 \leq i \leq 4;\ 1 \leq j \leq 3\}$ and

$$E(H_2) = \{x_{i,j}x_{i,j+1}, x_{1,2}x_{2,2}, x_{2,q}x_{3,q}, x_{1,s}x_{3,s}, x_{u,1}x_{u,3}: i = 2,4;\ 1 \leq j, q \leq 2;$$

$1 \leq s \leq 3;\ u = 2,4\}$. H_2 is decomposed into 2 copies of $K_{1,3}$ and 3 copies of $K_{1,2}$ as follows:

$\{x_{2,2}x_{1,2}, x_{2,2}x_{2,1}, x_{2,2}x_{4,2}\}$, $\{x_{2,3}x_{2,2}, x_{2,3}x_{2,1}, x_{3,2}x_{3,3}\}, \{x_{2,1}x_{3,1}, x_{2,1}x_{4,1}\}, \{x_{4,1}x_{4,2}, x_{4,1}x_{4,3}\}$

and $\{x_{4,3}x_{4,2}, x_{4,3}x_{2,3}\}$.

Remark 1.4.

Consider a graph H_3 with $V(H_3) = \{x_{i,j}: 1 \leq i \leq 3;\ 1 \leq j \leq 3\}$ and

$E(H_3) = \{x_{i,j}x_{i,j+1}, x_{1,l}x_{2,l}, x_{1,q}x_{3,q}: i = 1,3;\ 1 \leq j \leq 2;\ 2 \leq l, q \leq 3\}$. H_3 is decomposed into 2 copies of $K_{1,3}$ and 1 copy of $K_{1,2}$ as follows: $\{x_{1,2}x_{1,1}, x_{1,2}x_{2,2}, x_{1,2}x_{3,2}\}$,

$\{x_{1,3}x_{1,2}, x_{1,3}x_{2,3}, x_{1,3}x_{3,3}\}$ and $\{x_{3,2}x_{3,1}, x_{3,2}x_{3,3}\}$.

Theorem 1.5. *For any two positive integers m, n with $m \geq 3$, $n \geq 3$ and n is odd,*

$L_{2m} \square C_n$ is decomposed into p copies of $K_{1,3}$ and q copies of $K_{1,2}$ if and only if

$p + \frac{2q}{3} = 4mn$.

Proof:

Let $G = L_{2m} \square C_n$, where $V(G) = \{x_{i,j}: 1 \leq i \leq 2m;\ 1 \leq j \leq n\}$ and

$$E(G) = \{x_{i,j}x_{i,j+1}, x_{k,1}x_{k,1+n}, x_{p,q}x_{p+1,q}, x_{1,s}x_{m,s}, x_{u,v}x_{u+m,v}: 1 \leq i \leq 2m;$$

$$1 \leq j \leq n-1;\ 1 \leq k \leq 2m;\ 1 \leq p \leq m-1;\ 1 \leq q, s, v \leq n;\ 1 \leq u \leq m\}.$$

Suppose that $L_{2m} \square C_n$ is decomposed into p copies of $K_{1,3}$ and q copies of $K_{1,2}$.

Since $|E(G)| = 3p + 2q$, we have

$$3p + 2q = 3[n(m-1) - (5-m)] + 2[\frac{n(m+3) + 3(m-5)}{2}].$$

Therefore, $p + \frac{2q}{3} = 4mn$.

Conversely, suppose that $p + \frac{2q}{3} = 4mn$.

For $n = 3$, the induced subgraph $< \{x_{i,j-1}, x_{i,j}, x_{i,j+1}, x_{i+m-1,j-1}, x_{i+m-1,j}, x_{i+m-1,j+1}\} >$ together with the edges $\{x_{i-1,j}x_{i,j}, x_{i,j+1}x_{i+1,j+1}, x_{i,j-1}x_{i+1,j-1}\}$ where $j = 2$ forms $m - 3$ copies of H_2 $\forall\, i = 2, 3, \ldots, m-2$.

Thus, $E(L_{2m} \square C_3) = E(L_6 \square C_3) \cup [E(H_2) \cup \ldots \cup E(H_2)\ (m-3)\text{-}times]$.

Therefore, by Theorem 1.2., $L_6 \square C_3$ is decomposed into 8 copies of $K_{1,3}$ and 6 copies of $K_{1,2}$. Also, by Remark 1.3., H_2 is decomposed into $2(m-3)$ copies of $K_{1,3}$ and $3(m-3)$ copies of $K_{1,2}$. For $n \geq 5$, we prove this theorem by induction on n.

For $n = 5$, the induced subgraph $<$ $\{x_{i,4}, x_{i,5}, x_{m-1,4}, x_{m-1,5}, x_{m,4}, x_{m,5}, x_{m+1,4},$

$x_{m+1,5}, x_{2m-1,4}, x_{2m-1,5}, x_{2m,4}, x_{2m,5}\} >$ together with the edges $\{x_{i,3}x_{i,4}, x_{i+1,4}x_{i,4},$

$x_{i,5}x_{i+1,5}, x_{m-1,3}x_{m-1,4}, x_{m,3}x_{m,4}, x_{m+1,3}x_{m+1,4}, x_{2m-1,3}x_{2m-1,4}, x_{2m,3}x_{2m,4}\}$ forms H_1

for all $1 \leq i \leq 2m$. Also, the induced subgraph $< \{x_{i,4}, x_{i,5}, x_{i+m,4}, x_{i+m,5}\} >$ together with the edges $\{x_{i,3}x_{i,4}, x_{i,4}x_{i+1,4}, x_{i,5}x_{i+1,5}, x_{i+m,3}x_{i+m,4}\}$ forms $(m-3)$ copies of H_3 for all

$i = 2, 3, \ldots, m-2$.

Thus $E(L_{2m}\square C_5) = E(L_{2m}\square C_3) \cup E(H_1) \cup [E(H_3) \cup \ldots \cup E(H_3)\ (m-3)\text{-}times]$.

Assume that the theorem is true for $n-2$.

The induced subgraph $< \{x_{i,j}, x_{i,j+1}, x_{m-1,j}, x_{m-1,j+1}, x_{m,j}, x_{m,j+1}, x_{m+1,j}, x_{m+1,j+1},$

$x_{2m-1,j}, x_{2m-1,j+1}, x_{2m,j}, x_{2m,j+1}\} >$ together with the edges $\{x_{i,j-1}x_{i,j}, x_{i+1,j}x_{i,j},$

$x_{i,j+1}x_{i+1,j+1}, x_{m-1,j-1}x_{m-1,j}, x_{m,j-1}x_{m,j}, x_{m+1,j-1}x_{m+1,j}, x_{2m-1,j-1}x_{2m-1,j}, x_{2m,j-1}x_{2m,j}\}$

where $j = 6, 8, \ldots, n-1$ forms $H_1\ \forall\ 1 \leq i \leq 2m$.

Also, the induced subgraph $< \{x_{i,j}, x_{i,j+1}, x_{i+m,j}, x_{i+m,j+1}\} >$ together with the edges

$\{x_{i,j-1}x_{i,j}, x_{i,j}x_{i+1,j}, x_{i,j+1}x_{i+1,j+1}, x_{i+m,j-1}x_{i+m,j}\}$ where $j = 6, 8, \ldots, n-1$ forms 2 copies of $H_3\ \forall\ i = 2, 3, \ldots, m-2$.

Thus $E(L_{2m}\square C_n) = E(L_{2m}\square C_{n-2}) \cup E(H_1) \cup [E(H_3) \cup \ldots \cup E(H_3)\ (m-3)\text{-}times]$.

Therefore, by induction hypothesis, $L_{2m}\square\, C_{n-2}$ is decomposed into

$4(n-2)(m-1) + (5-m)$ copies of $K_{1,3}$ and $\frac{(n-2)(m+3)+3(m-5)}{2}$ copies of $K_{1,2}$. Also, by Remark 1.1., H_1 is decomposed into 4 copies of $K_{1,3}$ and 6 copies of $K_{1,2}$. And also, by Remark 1.4., H_3 is decomposed into $2(m-3)$ copies of $K_{1,3}$ and $(m-3)$ copies of $K_{1,2}$. Hence $L_{2m}\square C_n$ is decomposed into p copies of $K_{1,3}$ and q copies of $K_{1,2}$.

Theorem 1.6. *For any two positive integers m, n with $m = 3$, $n \geq 4$ and n is even,*

$L_{2m}\square C_n$ is decomposed into p copies of $K_{1,3}$, q copies of $K_{1,2}$ and 2 copies of P_4 if

and only if $p + \frac{2}{3}q = \frac{m(5n-1)}{3} - (n+1)$.

Proof:

Let $G = L_{2m}\square C_n$, where $V(G) = \{x_{i,j}: 1 \leq i \leq 2m;\ 1 \leq j \leq n\}$ and

$$E(G) = \{x_{i,j}x_{i,j+1}, x_{k,1}x_{k,1+n}, x_{p,q}x_{p+1,q}, x_{1,s}x_{m,s}, x_{u,v}x_{u+m,v}: 1 \leq i \leq 2m;\ 1 \leq j \leq n-1;\ 1 \leq k \leq 2m;\ 1 \leq p \leq m-1;\ 1 \leq q, s, v \leq n;\ 1 \leq u \leq m\}.$$

Suppose that $L_{2m}\square C_n$ is decomposed into p copies of $K_{1,3}$, q copies of $K_{1,2}$ and 2 copies

of P_4.

Since $|E(G)| = 3p + 2q + 6$, we have

$3p + 2q + 6 = 3[(m-1)(n+1)] + 2(mn-2m) + 6$.

$\Rightarrow 3p + 2q = 5mn - m - 3n - 3$.

Therefore, $p + \frac{2}{3}q = \frac{m(5n-1)}{3} - (n+1)$.

Conversely, suppose that $p + \frac{2}{3}q = \frac{m(5n-1)}{3} - (n+1)$.

For $n = 4$, the set of edges $\{x_{1,1}x_{1,2}, x_{1,1}x_{1,4}, x_{1,1}x_{2,1}\}, \{x_{1,2}x_{2,2}, x_{1,2}x_{1,3}, x_{1,2}x_{4,2}\},$ $\{x_{2,1}x_{3,1}, x_{2,1}x_{2,2}, x_{2,1}x_{5,1}\}, \{x_{2,2}x_{2,1}, x_{2,2}x_{3,2}, x_{2,2}x_{2,3}\}, \{x_{3,4}x_{2,4}, x_{3,4}x_{1,4}, x_{3,4}x_{3,3}\},$

$\{x_{3,1}x_{2,1}, x_{3,1}x_{1,1}, x_{3,1}x_{3,4}\}, \{x_{3,2}x_{3,3}, x_{3,2}x_{1,2}, x_{3,2}x_{6,2}\}, \{x_{3,3}x_{2,3}, x_{3,3}x_{1,3}, x_{3,3}x_{6,3}\},$

$\{x_{5,1}x_{2,1}, x_{5,1}x_{5,2}, x_{5,1}x_{5,4}\}, \{x_{4,4}x_{4,3}, x_{4,4}x_{4,1}, x_{4,4}x_{1,4}\}, \{x_{1,2}x_{2,2}, x_{1,2}x_{1,3}, x_{1,2}x_{4,2}\}$ and

$\{x_{6,4}x_{6,3}, x_{6,4}x_{6,1}, x_{6,4}x_{3,4}\}$ forms 10 copies of $K_{1,3}$. Also, the set of edges

$\{x_{1,4}x_{1,3}, x_{1,4}x_{2,4}\}, \{x_{2,4}x_{2,3}, x_{2,4}x_{2,1}\}, \{x_{3,1}x_{3,2}, x_{3,1}x_{6,1}\}, \{x_{6,2}x_{6,1}, x_{6,2}x_{6,3}\},$

$\{x_{5,4}x_{5,3}, x_{5,4}x_{2,4}\}$ and $\{x_{1,3}x_{2,3}, x_{1,3}x_{4,3}\}$ forms 6 copies of $K_{1,2}$. And also, the set of edges $\{x_{1,1}x_{4,1}, x_{4,1}x_{4,2}, x_{4,2}x_{4,3}\}$ and $\{x_{2,3}x_{5,3}, x_{5,3}x_{5,2}, x_{5,2}x_{2,2}\}$ forms 2 copies of P_4.

Therefore, $L_6\square C_4$ is decomposed into 10 copies of $K_{1,3}$, 6 copies of $K_{1,2}$ and 2 copies of P_4.

For $n \geq 6$, we prove this theorem by induction on n.

For $n = 6$, the induced subgraph $< \{x_{i,5}, x_{i,6}\} >$ together with the edge $x_{i,4}x_{i,5}$ forms $H_1\ \forall\ 1 \leq i \leq 2m$. Thus $E(L_{2m}\square C_6) = E(L_{2m}\square C_4) \cup E(H_1)$.

Assume that the theorem is true for $n-2$.

The induced subgraph $< \{x_{i,j}, x_{i,j+1}\} >$ together with the edge $x_{i,j-1}x_{i,j}$ where

$j = 7, 9, \ldots, n-1$ forms $H_1\ \forall\ 1 \leq i \leq 2m$. Thus $E(L_{2m}\square C_n) = E(L_{2m}\square C_{n-2}) \cup E(H_1)$.

Therefore, by induction hypothesis, $L_{2m}\square C_{n-2}$ is decomposed into $2(n-1)$ copies of $K_{1,3}$, $3(n-4)$ copies of $K_{1,2}$ and 2 copies of P_4. Also, by Remark 1.1., H_1 is decomposed into 4 copies of $K_{1,3}$ and 6 copies of $K_{1,2}$. Hence $L_{2m}\square C_n$ is decomposed into p copies of $K_{1,3}$, q copies of $K_{1,2}$ and 2 copies of P_4.

Remark 1.7.

Consider a graph H_4 with $V(H_4) = \{x_{i,j}: 1 \le i \le 3;\ 1 \le j \le 4\}$ and

$E(H_4) = \{x_{i,j}x_{i,j+1}, x_{1,l}x_{2,l}, x_{p,3}x_{p,6}: i,p = 1,3;\ 1 \le j \le 3;\ 1 \le l \le 4\}$. H_4 is decomposed into 2 copies of $K_{1,3}$ and 5 copies of $K_{1,2}$ as follows: $\{x_{1,4}x_{1,1}, x_{1,4}x_{1,3}, x_{1,4}x_{2,4}\}$, $\{x_{3,4}x_{3,1}, x_{3,4}x_{3,3}, x_{3,4}x_{1,4}\}$, $\{x_{1,1}x_{1,2}, x_{1,1}x_{2,1}\}$, $\{x_{1,2}x_{2,2}, x_{1,2}x_{3,2}\}$, $\{x_{1,3}x_{1,2}, x_{1,3}x_{2,3}\}$, $\{x_{4,1}x_{1,1}, x_{4,1}x_{4,2}\}$ and $\{x_{3,3}x_{3,2}, x_{3,3}x_{1,3}\}$.

Theorem 1.8. *For any two positive integers* m, n *with* $m \ge 3$, $n \ge 4$ *and* n *is even,*

$L_{2m}\square C_n$ *is decomposed into* p *copies of* $K_{1,3}$, q *copies of* $K_{1,2}$ *and* 2 *copies of* P_4 *if*

and only if $p + \frac{2}{3}q = \frac{2}{3}(2mn - 3)$.

Proof:

Let $G = L_{2m}\square C_n$, where $V(G) = \{x_{i,j}: 1 \le i \le 2m;\ 1 \le j \le n\}$ and

$$E(G) = \{x_{i,j}x_{i,j+1}, x_{k,1}x_{k,1+n}, x_{p,q}x_{p+1,q}, x_{1,s}x_{m,s}, x_{u,v}x_{u+m,v}: 1 \le i \le 2m;$$

$$1 \le j \le n-1;\ 1 \le k \le 2m;\ 1 \le p \le m-1;\ 1 \le q,s,v \le n;\ 1 \le u \le m\}.$$

Suppose that $L_{2m}\square C_n$ is decomposed into p copies of $K_{1,3}$, q copies of $K_{1,2}$ and 2 copies of P_4.

Since $|E(G)| = 3p + 2q + 6$, we have

$$3p + 2q + 6 = 3[n(m-1) + 2(m-4)] + 2[\frac{n(m+3)}{2} + 3(m-5)] + 6.$$

$$\Rightarrow 3p + 2q = 4mn - 6.$$

Therefore, $p + \frac{2}{3}q = \frac{2}{3}(2mn - 3)$.

Conversely, suppose that $p + \frac{2}{3}q = \frac{2}{3}(2mn - 3)$.

For $n = 4$, the induced subgraph $< \{x_{i-1,j-1}, x_{i-1,j}, x_{i-1,j+1}, x_{i-1,j+2}, x_{i+m-1,j-1}, x_{i+m-1,j}, x_{i+m-1,j+1}, x_{i+m-1,j+2}\} >$ together with the edges $\{x_{i-1,j-1}x_{i,j-1}, x_{i-1,j}x_{i,j}, x_{i-1,j+1}, x_{i,j+1}, x_{i-1,j+2}, x_{i,j+2}\}$ where $j = 2$ forms $m-3$ copies of $H_4\ \forall\ i = 3,4,\dots,m-1$.

Thus, $E(L_{2m}\square C_4) = E(L_6\square C_4) \cup [E(H_4) \cup \dots \cup E(H_4)\ (m-3)\text{-}times]$.

Therefore, by Theorem 1.6., $L_6\square C_4$ is decomposed into 10 copies of $K_{1,3}$, 6 copies of $K_{1,2}$

and 2 copies of P_4. Also, by Remark 1.7., H_4 is decomposed $2(m-3)$ copies of $K_{1,2}$ and

$5(m-3)$ copies of $K_{1,2}$. For $n \ge 6$, we prove this theorem by induction on n.

For $n = 6$, the induced subgraph $< \{x_{i,5}, x_{i,6}, x_{m-1,5}, x_{m-1,6}, x_{m,5}, x_{m,6}, x_{m+1,5}, x_{m+1,6}, x_{2m-1,5}, x_{2m-1,6}, x_{2m,5}, x_{2m,6}\} >$ together with the edges $\{x_{i,4}x_{i,5}, x_{i+1,5}x_{i,5}, x_{i,6}x_{i+1,6}, x_{m-1,4}x_{m-1,5}, x_{m,4}x_{m,5}, x_{m+1,4}x_{m+1,5}, x_{2m-1,4}x_{2m-1,5}, x_{2m,4}x_{2m,5}\}$ forms H_1 for all $1 \le i \le 2m$. Also, the induced subgraph $< \{x_{i,5}, x_{i,6}, x_{i+m,5}, x_{i+m,6}\} >$ together with the edges $\{x_{i,4}x_{i,5}, x_{i,5}x_{i+1,5}, x_{i,6}x_{i+1,6}, x_{i+m,4}x_{i+m,5}\}$ forms $(m-3)$ copies of H_3 for all

$i = 2,3,\dots,m-2$.

Thus $E(L_{2m}\square C_6) = E(L_{2m}\square C_4) \cup E(H_1) \cup [E(H_3) \cup \dots \cup E(H_3)\ (m-3)\text{-}times]$.

Assume that the theorem is true for $n-2$.

The induced subgraph $< \{x_{i,j}, x_{i,j+1}, x_{m-1,j}, x_{m-1,j+1}, x_{m,j}, x_{m,j+1}, x_{m+1,j}, x_{m+1,j+1}, x_{2m-1,j}, x_{2m-1,j+1}, x_{2m,j}, x_{2m,j+1}\} >$ together with the edges $\{x_{i,j-1}x_{i,j}, x_{i+1,j}x_{i,j}, x_{i,j+1}x_{i+1,j+1}, x_{m-1,j-1}x_{m-1,j}, x_{m,j-1}x_{m,j}, x_{m+1,j-1}x_{m+1,j}, x_{2m-1,j-1}x_{2m-1,j}, x_{2m,j-1}x_{2m,j}\}$

where $j = 7,9,\dots,n-1$ forms $H_1\ \forall\ 1 \le i \le 2m$.

Also, the induced subgraph $< \{x_{i,j}, x_{i,j+1}, x_{i+m,j}, x_{i+m,j+1}\} >$ together with the edges

$\{x_{i,j-1}x_{i,j}, x_{i,j}x_{i+1,j}, x_{i,j+1}x_{i+1,j+1}, x_{i+m,j-1}x_{i+m,j}\}$ where $j = 7,9,\dots,n-1$ forms $m-3$ copies of $H_3\ \forall\ i = 2,3,\dots,m-2$.

Thus $E(L_{2m}\square C_n) = E(L_{2m}\square C_{n-2}) \cup E(H_1) \cup [E(H_3) \cup \dots \cup E(H_3)\ (m-3)\text{-}times]$.

Therefore, by induction hypothesis, $L_{2m}\square C_{n-2}$ is decomposed into

$(n-2)(m-1) + 2(m-4)$ copies of $K_{1,3}$, $\frac{(m+3)(n-2)}{2} + 3(m-5)$ copies of $K_{1,2}$ and 2 copies of P_4. Also, by Remark 1.1., H_1 is decomposed into 4 copies of $K_{1,3}$ and 6 copies of $K_{1,2}$. And also, by Remark 1.4., H_3 is decomposed into $2(m-3)$ copies of $K_{1,3}$ and $(m-3)$

copies of $K_{1,2}$. Hence $L_{2m}\square C_n$ is decomposed into p copies of $K_{1,3}$, q copies of $K_{1,2}$ and 2 copies of P_4.

Theorem 1.9. *For any two positive integers m, n with $m \geq 3$, $n \geq 3$ and n is odd,*

$L_{2m} \circledcirc C_n$ is decomposed into p copies of $K_{1,3} + e$ and q copies of $K_{1,3}$ if and only if

$2p + \frac{3}{2}q = m(n+1)$.

Proof:

Let $G = L_{2m} \circledcirc C_n$.

Let $V(L_{2m}) = \{u_i, v_i : 1 \leq i \leq m\}$, where u_i is the pendant vertex adjacent to v_i.

Let $V(G) = \{u_i, v_i, u_{ij'}, v_{ij'} : 1 \leq i \leq m, 1 \leq j \leq n\}$, where $u_{i1'}, u_{i2'}, \ldots, u_{in'}$ are the vertices in the copy of the cycle corresponding to each u_i and $v_{i1'}, v_{i2'}, \ldots, v_{in'}$ are the vertices in the copy of the cycle corresponding to each v_i.

Suppose that G is decomposed into p copies of $K_{1,3} + e$ and q copies of $K_{1,3}$.

Since $|E(G)| = 4p + 3q$, we have

$4p + 3q = 4(mn - m) + 3(2m)$.

$\Rightarrow 4p + 3q = 4mn + 2m$.

Therefore, $2p + \frac{3}{2}q = m(n+1)$.

Conversely, suppose that $2p + \frac{3}{2}q = m(n+1)$.

Now, the induced subgraph $< \{u_{i1'}, u_{i2'}, u_i, v_i\} > \cong K_{1,3} + e \; \forall \; 1 \leq i \leq m$.

The induced subgraph $< \{u_{i(j-1)'}, u_{ij'}, u_i\} >$ together with the edge $u_{i(j-2)'} \, u_{i(j-1)'}$, where $j = 4, 6, \ldots, n-1$ forms $K_{1,3} + e \; \forall \; 1 \leq i \leq m$.

The induced subgraph $< \{u_{i(n-1)'}u_{in'}, u_{in'}u_{i1'}, u_{in'}u_i\} > \cong K_{1,3} \; \forall \; 1 \leq i \leq m$.

Also, the induced subgraph $< \{v_{(i-1)1'}, v_{(i-1)2'}, v_{i-1}, v_i\} > \cong K_{1,3} + e \; \forall \; 2 \leq i \leq m$.

The induced subgraph $< \{v_{i(j-1)'}, v_{ij'}, v_i\} >$ together with the edge $v_{i(j-2)'} \, v_{i(j-1)'}$, where $j = 4, 6, \ldots, n-1$ forms $K_{1,3} + e \; \forall \; 1 \leq i \leq m$.

The induced subgraph $< \{v_{i(n-1)'}v_{in'}, v_{in'}v_{i1'}, v_{in'}v_i\} > \cong K_{1,3} \; \forall \; 1 \leq i \leq m$.

And also, the induced subgraph $< \{v_{m1'}, v_{m2'}, v_m, v_1\} > \cong K_{1,3} + e$.

The induced subgraph $< \{v_{m(j-1)'}, v_{mj'}, v_m\} >$ together with the edge $v_{m(j-2)'} \, v_{m(j-1)'}$, where $j = 4, 6, \ldots, n-1$ forms $K_{1,3} + e$.

The induced subgraph $< \{v_{m(n-1)'}v_{mn'}, v_{mn'}v_{m1'}, v_{mn'}v_m\} > \cong K_{1,3}$.

Thus $E(G) = [E(K_{1,3} + e) \cup E(K_{1,3} + e) \cup \ldots \cup E(K_{1,3} + e) \; (mn - m)\text{ - }times]$

$\cup \; [E(K_{1,3}) \cup E(K_{1,3}) \cup \ldots \cup E(K_{1,3}) \; 2m\text{ - }times]$.

Hence $L_{2m} \circledcirc C_n$ is decomposed into$p$copies of $K_{1,3} + e$and q copies of $K_{1,3}$.

Corollary 1.10. *For any two positive integers m, n with $m \geq 3$, $n \geq 3$ and n is odd,*

$H_m \circledcirc C_n$ is decomposed into p copies of $K_{1,3} + e$, q copies of $K_{1,3}$ and one copies of S_m

if and only if $p + \frac{3q}{4} = \frac{(2m+1)(2n+1)}{4}$.

Theorem 1.11. *For any two positive integers m, n with $m \geq 3$, $n \geq 4$ and n is even,*

$L_{2m} \circledcirc C_n$ is decomposed into p copies of $K_{1,3} + e$and q copies of $K_{1,2}$ if and only if

$2p + q = m(2n+1)$.

Proof:

Let $G = L_{2m} \circledcirc C_n$.

Let $V(L_{2m}) = \{u_i, v_i : 1 \leq i \leq m\}$, where u_i is the pendant vertex adjacent to v_i.

Let $V(G) = \{u_i, v_i, u_{ij'}, v_{ij'} : 1 \leq i \leq m, 1 \leq j \leq n\}$, where $u_{i1'}, u_{i2'}, \ldots, u_{in'}$ are the vertices in the copy of the cycle corresponding to each u_i and $v_{i1'}, v_{i2'}, \ldots, v_{in'}$ are the vertices in the copy of the cycle corresponding to each v_i.

Suppose that G is decomposed into p copies of $K_{1,3} + e$ and q copies of $K_{1,2}$.

Since $|E(G)| = 4p + 2q$, we have

$4p + 2q = 4mn + 2m$.

Therefore, $2p + q = m(2n+1)$.

Conversely, suppose that $2p + q = m(2n+1)$.

Now, the induced subgraph $< \{u_{i(j-1)'}, u_{ij'}, u_i\} >$ together with the edge $u_{ij'} \, u_{i(j+1)'}$, where $j = 2, 4, \ldots, n-2$ forms $K_{1,3} + e \; \forall \; 1 \leq i \leq m$.

The induced subgraph $< \{v_{i(j-1)'}, v_{ij'}, v_i\} >$ together with the edge $v_{ij'} \, v_{i(j+1)'}$, where

$j = 2, 4, \ldots, n-2$ forms $K_{1,3} + e \; \forall \; 1 \leq i \leq m$.

Also, the induced subgraph $< \{u_i v_i, v_i v_{i+1}\} > \cong K_{1,2} \; \forall \; 1 \leq i \leq m-1$.

And also, the induced subgraph $< \{u_m v_m, v_m v_1\} > \cong K_{1,2}$.

Thus $E(G) = [E(K_{1,3}+e) \cup E(K_{1,3}+e) \cup \ldots \cup E(K_{1,3}+e)\ mn\text{-}times]$

$$\cup\ [E(K_{1,2}) \cup E(K_{1,2}) \cup \ldots \cup E(K_{1,2})\ m\text{-}times].$$

Hence $L_{2m} \circledcirc C_n$ is decomposed into p copies of $K_{1,3}+e$ and q copies of $K_{1,2}$.

Corollary 1.12. *For any two positive integers* m, n *with* $m \geq 3$, $n \geq 4$ *and* n *is even,*

$H_m \circledcirc C_n$ *is decomposed into* p *copies of* $K_{1,3}+e$, q *copies of* $K_{1,2}$ *and one copy of* S_m

if and only if $p + \frac{q}{2} = \frac{m(2n+1)+n}{2}$.

Applications:

The cartesian product of graphs is an important mathematical tool used to model drug delivery systems involving interacting factors. Since several independent biological or pharmaceutical variables interact simultaneously, cartesian product of graphs help to represent these complex interactions in a structural mathematical form. Hence this powerful mathematical framework can improve drug design, therapeutic efficiency and personalized medicine.

The corona product of graphs provide an effective mathematical framework for modelling hierarchical and multi-functional drug delivery systems where a central unit interacts with multiple similar sub-units. It is particularly useful in studying nanoparticle-based delivery, targeted receptor interactions, layered drug release mechanisms and combination therapies. In pharmaceutical and biomedical systems, many drug delivery mechanisms follow this hierarchical structure, making the corona product an effective tool for analysis and design..

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